THE CONTACT OF A CUBIC SURFACE WITH AN ANALYTIC SURFACE*

BY

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I. Introduction

The projective differential geometry of a surface in the neighborhood of one of its points has been enriched by the consideration of various quadrics covariant to the surface. Among these are the quadrics of Darboux, the quadric of Lie, and the canonical quadric of Wilczynski. All of these are members of the three-parameter family of quadrics having contact of the second order with the surface at the point considered. There is no non-singular quadric having contact of the third order at an ordinary point of an unrestricted surface. In fact, if there exists a non-singular quadric having contact of the third order, at a general point of a surface, then the surface itself is a quadric, and there is a pencil of quadrics having contact of the third order at each point. All of these pencils contain, of course, the surface itself.

It is the purpose of this paper to investigate the contact of a *cubic surface* with an analytic surface and to determine *necessary and sufficient conditions that a surface be a cubic.* Since a cubic is determined by nineteen points, and since it is necessary to impose $(n+1)(n+2)/2$ conditions to make an algebraic surface have contact of order n with an analytic surface, it follows that there is a four-parameter family of cubics having contact of order four at a point of an analytic surface. There is no non-composite cubic with contact of order five at a general point of a surface, unless the surface is restricted to be itself a cubic. There is a pencil of cubics with contact of the fifth order at each point of a cubic surface. These remarks suffice to indicate the trend of the following investigation.

II. Power series expansions

Let the four homogeneous coördinates $y^{(1)}, \cdots, y^{(4)}$ of a general point y on a non-degenerate non-ruled surface S be given as analytic functions of two independent variables u, v. If the curves $u=$const. and $v=$const. are the asymptotics, then the four functions y are solutions of a system of dif-

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ferential equations which can be reduced to Wilczynski's canonical form,

$$y_{uu} + 2by_v + fy = 0, \qquad y_{vv} + 2a'y_u + gy = 0, \tag{1}$$

whose coefficients are functions of u, v satisfying the following conditions of complete integrability:

$$\begin{aligned} a'_{uu} + g_u + 2ba'_v + 4a'b_v &= 0, \\ b_{vv} + f_v + 2a'b_u + 4ba'_u &= 0, \\ g_{uu} + 4gb_v + 2bg_v &= f_{vv} + 4fa'_u + 2a'f_u. \end{aligned} \tag{2}$$

The derivatives of y of all orders can be expressed uniquely as linear combinations of y, y_u, y_v, y_{uv}. We shall need the derivatives up to those of the fifth order inclusive, but we shall not need the coefficient of y for the derivatives of the fourth order, and we shall need only the coefficient of y_{uv} for the fifth order. The needed formulas will now be written, ***with the accent omitted from the a'***, for purposes of symmetry and simplicity:

$$\begin{aligned} y_{uuu} &= -f_u y - fy_u - 2b_u y_v - 2by_{uv}, \\ y_{uuv} &= (2bg - f_v)y + 4aby_u - (f + 2b_v)y_v + 0, \\ y_{uvv} &= (2afy - g_u)y - (g + 2a_u)y_u + 4aby_v + 0, \\ y_{vvv} &= -g_v y - 2a_v y_u - gy_v - 2ay_{uv}; \end{aligned}$$

$$\begin{aligned} y_{uuuu} &= \cdots - 2(f_u + 4ab^2)y_u + 2(2bf + 2bb_v - b_{uu})y_v - 4b_u y_{uv}, \\ y_{uuuv} &= \cdots + (4a_u b + 4ab_u + 2bg - f_v)y_u - (f_u + 2b_{uv} + 8ab^2)y_v \\ &\quad - (f + 2b_v)y_{uv}, \\ y_{uuvv} &= \cdots + 2(af + 4ab_v + 2a_v b)y_u + 2(bg + 4a_u b + 2ab_u)y_v + 4aby_{uv}, \\ y_{uvvv} &= \cdots - (g_v + 2a_{uv} + 8a^2b)y_u + (4a_v b + 4ab_v + 2af - g_u)y_v \\ &\quad - (g + 2a_u)y_{uv}, \\ y_{vvvv} &= \cdots + 2(2ag + 2a_u - a_{vv})y_u - 2(g_v + 4a^2b)y_v - 4a_v y_{uv}; \end{aligned} \tag{3}$$

$$\begin{aligned} y_{uuuuu} &= \cdots + 2(2bf + 2bb_v - 3b_{uu})y_{uv}, \\ y_{uuuuv} &= \cdots - 2(f_u + 4ab^2 + 2b_{uv})y_{uv}, \\ y_{uuuvv} &= \cdots + 2(bg + 6a_u b + 4ab_u)y_{uv}, \\ y_{uuvvv} &= \cdots + 2(af + 6ab_v + 4a_v b)y_{uv}, \\ y_{uvvvv} &= \cdots - 2(g_v + 4a^2b + 2a_{uv})y_{uv}, \\ y_{vvvvv} &= \cdots + 2(2ag + 2aa_u - 3a_{vv})y_{uv}. \end{aligned}$$

If the points y, y_u, y_v, y_{uv} are used as the vertices of a local tetrahedron of reference, with unit point suitably chosen, then the coördinates of a point

x on S near the point y may be easily calculated as power series in the increments Δu and Δv corresponding to displacement from P_y to P_x on S. We shall now write these series to as many terms as needed, using $\epsilon=\Delta u$, $\eta=\Delta v$:

$$
\begin{aligned}
x_1 &= 1 - \frac{1}{2}(f\epsilon^2 + g\eta^2) - \frac{1}{6}f_u\epsilon^3 + \frac{1}{2}(2bg - f_v)\epsilon^2\eta \\
&\quad + \frac{1}{2}(2af - g_u)\epsilon\eta^2 - \frac{1}{6}g_v\eta^3 + \cdots, \\
x_2 &= \epsilon - a\eta^2 - \frac{1}{6}f\epsilon^3 + 2ab\epsilon^2\eta - \frac{1}{2}(g + 2a_u)\epsilon\eta^2 - \frac{1}{3}a_v\eta^3 \\
&\quad - \frac{1}{12}(f_u + 4ab^2)\epsilon^4 + \frac{1}{6}(4a_ub + 4ab_u + 2bg - f_v)\epsilon^3\eta + \frac{1}{2}(af + 4ab_v \\
&\quad + 2a_vb)\epsilon^2\eta^2 - \frac{1}{6}(g_v + 2a_{uv} + 8a^2b)\epsilon\eta^3 \\
&\quad + \frac{1}{6}\left(ag + aa_u - \frac{1}{2}a_{vv}\right)\eta^4 + \cdots, \\
x_3 &= \eta - b\epsilon^2 - \frac{1}{3}b_u\epsilon^3 - \frac{1}{2}(f + 2b_v)\epsilon^2\eta + 2ab\epsilon\eta^2 - \frac{1}{6}g\eta^3 \qquad (4)\\
&\quad + \frac{1}{6}\left(bf + bb_v - \frac{1}{2}b_{uu}\right)\epsilon^4 - \frac{1}{6}(f_u + 2b_{uv} + 8ab^2)\epsilon^3\eta \\
&\quad + \frac{1}{2}(bg + 4a_ub + 2ab_u)\epsilon^2\eta^2 + \frac{1}{6}(4a_vb + 4ab_v + 2af - g_u)\epsilon\eta^3 \\
&\quad - \frac{1}{12}(g_v + 4a^2b)\eta^4 + \cdots, \\
x_4 &= \epsilon\eta - \frac{1}{3}(b\epsilon^3 + a\eta^3) - \frac{1}{6}b_u\epsilon^4 - \frac{1}{6}(f + 2b_v)\epsilon^3\eta + ab\epsilon^2\eta^2 \\
&\quad - \frac{1}{6}(g + 2a_u)\epsilon\eta^3 - \frac{1}{6}a_v\eta^4 + \frac{1}{60}(2bf + 2bb_v - 3b_{uu})\epsilon^5 \\
&\quad - \frac{1}{12}(f_u + 2b_{uv} - 4ab^2)\epsilon^4\eta + \frac{1}{6}(bg + 6a_ub + 4ab_u)\epsilon^3\eta^2 \\
&\quad + \frac{1}{6}(af + 4a_vb + 6ab_v)\epsilon^2\eta^3 - \frac{1}{12}(g_v + 2a_{uv} + 4a^2b)\epsilon\eta^4 \\
&\quad + \frac{1}{60}(2ag + 2aa_u - 3a_{vv})\eta^5 + \cdots .
\end{aligned}
$$

III. Cubics with various contacts

Let us write the equation of a cubic surface in the form

$$a_{111}x_1^3 + \cdots + a_{234}x_2x_3x_4 = 0, \tag{5}$$

the subscripts in the case of each coefficient indicating the term to which it belongs. This cubic passes through the point y if, and only if, $a_{111}=0$. It has first order contact with the surface S at the point y if also $a_{112}=a_{113}=0$. It has second order contact if also $a_{122}=a_{133}=a_{114}+a_{123}=0$, as may be verified by observing that these conditions are necessary and sufficient that the power series for the twenty possible cubic combinations of the x_i as obtained from (4) satisfy equation (5) identically in ϵ and η up to, and including, terms of the second order. Similarly, conditions necessary and sufficient for third order contact are found to be, in addition to those just mentioned,

$$a_{222} = \frac{2}{3}ba_{123}, \quad a_{333} = \frac{2}{3}aa_{123}, \quad a_{124} + a_{223} = 0, \quad a_{134} + a_{233} = 0.$$

The conditions for fourth order contact may be reduced to

$$a_{223} = -\frac{1}{4}\frac{b_u}{b}a_{123}, \quad a_{233} = -\frac{1}{4}\frac{a_v}{a}a_{123}, \quad a_{234} + a_{144} = 0,$$

$$a_{224} = \left(\frac{2}{3}b_v - \frac{1}{6}b\frac{a_v}{a}\right)a_{123}, \quad a_{334} = \left(\frac{2}{3}a_u - \frac{1}{6}a\frac{b_u}{b}\right)a_{123},$$

in addition to the foregoing, so that *the equation of the most general cubic having contact of the fourth order with the surface S at the point y may be written in the form*

$$\begin{aligned} a_{123}\Big[\frac{2}{3}(bx_2^3 + ax_3^3) &+ (x_2x_3 - x_1x_4)\left(x_1 - \frac{1}{4}\frac{b_u}{b}x_2 - \frac{1}{4}\frac{a_v}{a}x_3\right) \\ &+ \left(\frac{2}{3}b_v - \frac{1}{6}b\frac{a_v}{a}\right)x_2^2x_4 + \left(\frac{2}{3}a_u - \frac{1}{6}a\frac{b_u}{b}\right)x_3^2x_4\Big] \\ &+ a_{144}(x_1x_4 - x_2x_3)x_4 + a_{244}x_2x_4^2 + a_{344}x_3x_4^2 + a_{444}x_4^3 = 0. \end{aligned} \tag{6}$$

When the terms of the fifth order are substituted from the power series for the cubic combinations of the x_i into equation (6), the result may be arranged in the form

$$
(7)\qquad \begin{aligned}
&\left\{\frac{2}{15}ba_{123}\left[f-\frac{2}{3}b_v-\frac{5}{6}b\frac{a_v}{a}-\frac{1}{4}\frac{b_{uu}}{b}+\frac{5}{16}\left(\frac{b_u}{b}\right)^2\right]\right\}\epsilon^5\\
&+\left\{\frac{2}{3}ba_{144}-\frac{1}{6}ba_{123}\left[\frac{b_{uv}}{b}-\frac{b_ub_v}{b^2}-16ab-\frac{1}{4}\frac{a_v}{a}\frac{b_u}{b}\right]\right\}\epsilon^4\eta\\
&+\left\{a_{244}-\frac{2}{3}ba_{123}\left[g+\frac{7}{6}a\frac{b_u}{b}+\frac{10}{3}a_u+\frac{1}{2}\frac{b_{vv}}{b}-\frac{1}{4}\frac{a_v}{a}\frac{b_v}{b}\right]\right\}\epsilon^3\eta^2\\
&+\cdots=0,
\end{aligned}
$$

the omitted terms being obtained from those written by interchange of a and b, f and g, u and v, 2 and 3, ϵ and η. If this equation is satisfied identically in ϵ and η without restriction on the surface S, the cubic (6) reduces to $x_4^3=0$, that is, the tangent plane counted three times. *If the cubic* (6) *is restricted to be non-composite, then* $a_{123}\neq 0$, *and equation* (7) *will be satisfied identically in* ϵ *and* η *provided that the surface S is restricted by the three conditions*

$$
(8)\qquad \begin{aligned}
&\frac{\partial^2}{\partial u\partial v}\log\frac{a}{b}=0,\\
&f=\frac{2}{3}b_v+\frac{5}{6}b\frac{a_v}{a}+\frac{1}{4}\frac{b_{uu}}{b}-\frac{5}{16}\left(\frac{b_u}{b}\right)^2,\\
&g=\frac{2}{3}a_u+\frac{5}{6}a\frac{b_u}{b}+\frac{1}{4}\frac{a_{vv}}{a}-\frac{5}{16}\left(\frac{a_v}{a}\right)^2,
\end{aligned}
$$

and provided that the coefficients of the cubic (6) *satisfy three conditions which can easily be written down. Then all the cubics represented by the following equation have contact of the fifth order with the surface S*:

$$
(9)\qquad \begin{aligned}
&\frac{2}{3}(bx_2^3+ax_3^3)+(x_2x_3-x_1x_4)\left(x_1-\frac{1}{4}\frac{b_u}{b}x_2-\frac{1}{4}\frac{a_v}{a}x_3\right.\\
&\left.-\frac{1}{4}\left[\frac{\partial}{\partial v\partial u}\log b-16ab-\frac{1}{4}\frac{a_v}{a}\frac{b_u}{b}\right]x_4\right)\\
&+\left(\frac{2}{3}b_v-\frac{1}{6}b\frac{a_v}{a}\right)x_2^2x_4+\left(\frac{2}{3}a_u-\frac{1}{6}a\frac{b_u}{b}\right)x_3^2x_4\\
&+\frac{2}{3}b\left(g+\frac{7}{6}a\frac{b_u}{b}+\frac{10}{3}a_u+\frac{1}{2}\frac{b_{vv}}{b}-\frac{1}{4}\frac{a_v}{a}\frac{b_v}{b}\right)x_4^2x_2\\
&+\frac{2}{3}a\left(f+\frac{7}{6}b\frac{a_v}{a}+\frac{10}{3}b_v+\frac{1}{2}\frac{a_{uu}}{a}-\frac{1}{4}\frac{a_u}{a}\frac{b_u}{b}\right)x_4^2x_3+\lambda x_4^3=0,
\end{aligned}
$$

where λ *is an arbitrary function of* u, v.

One of my students, Miss Selma Learman, in her master's thesis (Chicago, 1926) has obtained conditions (8) as necessary conditions that a surface be a cubic. The lengthy and tedious calculations involved in reaching them were performed by both of us independently and checked by comparison.

IV. Simplification of conditions

The first of conditions (8) expresses that the surface S is of the kind called* by Fubini *isothermally asymptotic* (isotermo-asintotica). It is on such a surface that the directrix curves of Wilczynski form a conjugate net. Since this condition implies that the ratio of a to b is a function of u alone times a function of v alone, it follows that it is possible, by means of a transformation which does not disturb the canonical form of equations (1), to reduce this ratio to unity so that we shall have $a=b$. We shall hereafter suppose, unless indicated to the contrary, that this transformation has been made. Then the first of conditions (8) is satisfied identically, and the other two become

$$\begin{aligned} f &= \frac{3}{2}b_v + \frac{1}{4}\frac{b_{uu}}{b} - \frac{5}{16}\left(\frac{b_u}{b}\right)^2, \\ g &= \frac{3}{2}b_u + \frac{1}{4}\frac{b_{vv}}{b} - \frac{5}{16}\left(\frac{b_v}{b}\right)^2. \end{aligned} \tag{10}$$

Equation (9) becomes, in consequence of these conditions,

$$\begin{aligned} &\frac{2}{3}b(x_2^3 + x_3^3) + (x_2x_3 - x_1x_4)\left(x_1 - \frac{1}{4}\frac{b_u}{b}x_2 - \frac{1}{4}\frac{b_v}{b}x_3\right. \\ &\qquad \left. - \frac{1}{4}\left[\frac{\partial^2}{\partial u \partial v}\log b - 16b^2 - \frac{1}{4}\frac{b_ub_v}{b^2}\right]x_4\right) \\ &+ \frac{1}{2}b_vx_2^2x_4 + \frac{1}{2}b_ux_3^2x_4 + \frac{2}{3}b\left[6b_u + \frac{3}{4}\frac{b_{uu}}{b} - \frac{9}{16}\left(\frac{b_v}{b}\right)^2\right]x_4^2x_2 \\ &+ \frac{2}{3}b\left[6b_v + \frac{3}{4}\frac{b_{vv}}{b} - \frac{9}{16}\left(\frac{b_u}{b}\right)^2\right]x_4^2x_3 + \lambda x_4^3 = 0. \end{aligned} \tag{11}$$

By means of the relation $a=b$ and conditions (10), it is possible to reduce the first two of the integrability conditions (2) to the form

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* Fubini and Čech, *Geometria Proiettiva Differenziale*, Bologna, Zanichelli, 1926, vol. I, p. 115.

$$
(12) \qquad
\begin{aligned}
\left[\frac{\partial^2}{\partial u \partial v}\log b + 12b^2\right]_v &= \left[\frac{1}{4}\left(\frac{b_v}{b}\right)^2 - 10b_u\right]_u, \\
\left[\frac{\partial^2}{\partial u \partial v}\log b + 12b^2\right]_u &= \left[\frac{1}{4}\left(\frac{b_u}{b}\right)^2 - 10b_v\right]_v.
\end{aligned}
$$

Differentiating the first of these equations with respect to u and the second with respect to v, and subtracting, we obtain, after some reductions, the symmetric relation

$$
(13) \qquad 2bb_{uuu} - b_u b_{uu} = 2bb_{vvv} - b_v b_{vv}.
$$

Direct calculation now suffices to show that *the third of integrability conditions* (2) *is a consequence of the other two. Therefore, a surface S which, at each of of its points, admits a cubic surface with contact of order five, is an integrating surface of the system of differential equations*

$$
(14) \qquad
\begin{aligned}
y_{uu} + 2by_v + \left[\frac{3}{2}b_v + \frac{1}{4}\frac{b_{uu}}{b} - \frac{5}{16}\left(\frac{b_u}{b}\right)^2\right]y &= 0, \\
y_{vv} + 2by_u + \left[\frac{3}{2}b_u + \frac{1}{4}\frac{b_{vv}}{b} - \frac{5}{16}\left(\frac{b_v}{b}\right)^2\right]y &= 0,
\end{aligned}
$$

in which b is a solution of equations (12).

A particular solution of equations (12) is $b=\text{const}$. Then system (14) becomes

$$
y_{uu} + 2by_v = 0, \quad y_{vv} + 2by_u = 0,
$$

and four particular solutions thereof are

$$
y^{(1)} = 1, \; y^{(2)} = e^{-2b(u+v)}, \; y^{(3)} = e^{-2b(\omega u + \omega^2 v)}, \; y^{(4)} = e^{2b(\omega^2 u + \omega v)},
$$

where ω is a complex cube root of unity. If u and v are eliminated from these equations we find that the algebraic equation of this integrating surface is

$$
y^{(2)}y^{(3)}y^{(4)} = (y^{(1)})^3,
$$

or, in non-homogeneous coördinates, $xyz=1$. This is the cubic surface which appears in the projective theory as the analogue of the sphere, in the sense that the projective lines of curvature on this surface are indeterminate, the projective normals all passing through a fixed point.

V. The pencil of cubics with contact of the fifth order

At each point of a surface S defined by equations (14) subject to the conditions of integrability (12) there is a pencil of cubic surfaces with contact of the fifth order, represented by equation (11). Let us consider a particular

cubic of this family. As the point y varies over the surface S this cubic has an envelope, part of which, certainly, is the surface S. We shall proceed to find the points of contact of the cubic with its envelope.

If $x_1, \cdots, x_4$ are the coördinates of a point x referred to the local tetrahedron y, y_u, y_v, y_{uv} of a surface S at a point y and if $\xi_1, \cdots, \xi_4$ are the coördinates of the same point x referred to the corresponding local tetrahedron at a point on S near P_y corresponding to increments ϵ and η of u and v respectively, then the equations of transformation between these two tetrahedrons are, in general,

$$
\begin{aligned}
\xi_1 &= x_1 + f\epsilon x_2 + g\eta x_3 + [(f_v - 2bg)\epsilon + (g_u - 2af)\eta]x_4 + \cdots, \\
\xi_2 &= -\epsilon x_1 + x_2 + 2a\eta x_3 + [-4ab\epsilon + (g + 2a_u)\eta]x_4 + \cdots, \\
\xi_3 &= -\eta x_1 + 2b\epsilon x_2 + x_3 + [(f + 2b_v)\epsilon - 4ab\eta]x_4 + \cdots, \\
\xi_4 &= 0 - \eta x_2 - \epsilon x_3 + x_4 + \cdots.
\end{aligned} \tag{15}
$$

Making $a=b$ we may find for the fifteen cubic combinations of the coördinates which appear in equation (11) the formulas of transformation, corresponding to displacement along the asymptotic $v=\text{const}$. If we write the equation of the cubic corresponding to (11) at a point on S near P_y but referred, by means of these formulas of transformation, to the local tetrahedron at P_y; if we then subtract from this equation the equation (11); if we divide the result by ϵ and then let ϵ approach zero, we find the equation of a surface which intersects the cubic (11) in the points at which it touches its envelope as u alone varies. This equation has precisely the same fifteen cubic combinations of the coördinates which appear in equation (11). Moreover, thirteen of these have coefficients which are independent of λ and are precisely the same as the corresponding coefficients of (11), except perhaps for a common factor. The two terms whose coefficients contain λ are the terms in $x_3x_4^2$ and x_4^3. Indicating, therefore, the sum of the other thirteen terms of equation (11) by ϕ, we may write the new equation which we have obtained in the form

$$
\frac{1}{4}\frac{b_u}{b}\phi + (A^{(u)} - 3\lambda)x_3x_4^2 + (B^{(u)} + \lambda_u)x_4^3 = 0, \tag{16}
$$

where

$$
\begin{aligned}
A^{(u)} = {} & \frac{2}{3}\frac{\partial}{\partial u}\left\{ b\left[f + \frac{9}{2}b_v + \frac{1}{2}\frac{b_{uu}}{b} - \frac{1}{4}\left(\frac{b_u}{b}\right)^2 \right]\right\} + b_u(f + 2b_v) \\
& - \frac{1}{4}\frac{b_v}{b}(2bg - f_v) + b^2\left(\frac{\partial^2}{\partial u \partial v}\log b - 16b^2 - \frac{1}{4}\frac{b_ub_v}{b^2}\right),
\end{aligned} \tag{17}
$$

$$B^{(u)} = -\frac{1}{4}(2bg - f_v)\left(\frac{\partial^2}{\partial u \partial v}\log b - 16b^2 - \frac{1}{4}\frac{b_u b_v}{b^2}\right)$$
$$- \frac{8}{3}b^3\left[g + \frac{9}{2}b_u + \frac{1}{2}\frac{b_{vv}}{b} - \frac{1}{4}\left(\frac{b_v}{b}\right)^2\right]$$
$$+ \frac{2}{3}b(f + 2b_v)\left[f + \frac{9}{2}b_v + \frac{1}{2}\frac{b_{uu}}{b} - \frac{1}{4}\left(\frac{b_u}{b}\right)^2\right],$$

and f, g have the values given by equations (10). Taking a linear combination of equations (11) and (16), we find that, *as u alone varies, the cubic* (11) *touches its envelope, besides at the point y, also at points which lie in the plane*

$$\left\{A^{(u)} - 3\lambda - \frac{1}{6}b_u\left[6b_v + \frac{3}{4}\frac{b_{uu}}{b} - \frac{9}{16}\left(\frac{b_u}{b}\right)^2\right]\right\}x_3 + \left(B^{(u)} + \lambda_u - \frac{1}{4}\frac{b_u}{b}\lambda\right)x_4 = 0. \tag{18}$$

Similarly, *as v alone varies, the cubic* (11) *touches its envelope, besides at the point y, also at points which lie in the plane*

$$\left\{A^{(v)} - 3\lambda - \frac{1}{6}b_v\left[6b_u + \frac{3}{4}\frac{b_{vv}}{b} - \frac{9}{16}\left(\frac{b_v}{b}\right)^2\right]\right\}x_2 + \left(B^{(v)} + \lambda_v - \frac{1}{4}\frac{b_v}{b}\right)x_4 = 0, \tag{19}$$

where $A^{(v)}$ and $B^{(v)}$ are obtained from $A^{(u)}$ and $B^{(u)}$, respectively, by interchanging u and v. The line represented by equations (18) *and* (19) *pierces the cubic* (11) *in the points where it touches its envelope, u and v varying.*

At the close of §IV we found that for a particular value of b, namely $b = \text{const.}$, the surface S defined by equations (14) subject to (12) is a cubic surface. We are now in a position to show that *every surface defined by equations* (14) *and* (12) *is a cubic.* Such a surface is a cubic in case there exists a function λ such that the cubic (11) is independent of u, v, or is the same at every point of the surface; in this case the cubic (11) is the surface S itself. Such a function λ must be a common solution of the four equations obtained by setting equal to zero the four coefficients in equations (18) and (19). *These four equations have a common solution λ, which is given by the formula*

$$
\begin{aligned}
\lambda = \frac{1}{6} b_{uuu} - \frac{5}{24}\frac{b_u b_{uu}}{b} + \frac{5}{96} b_u \left(\frac{b_u}{b}\right)^2 + \frac{5}{3} b b_{uv} + b_u b_v \\
- \frac{1}{8}\frac{b_v b_{vv}}{b} + \frac{5}{96} b_v \left(\frac{b_v}{b}\right)^2 - \frac{16}{3} b^4.
\end{aligned} \tag{20}
$$

That this function is symmetric with respect to u and v is easily verified by means of equations (13).

We conclude that *conditions* (8) *are necessary and sufficient that the integrating surfaces of system* (1) *be cubics.*

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DYNAMICAL SPACE-TIMES WHICH CONTAIN A CONFORMAL EUCLIDEAN 3-SPACE*

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Einstein's theory of gravitation requires that a physical universe be a four-dimensional manifold whose line element is determined by an invariant quadratic differential form whose coefficients $g_{\mu\nu}$ satisfy the ten differential equations

$$G_{\mu\nu} = \lambda g_{\mu\nu} \tag{0.1}$$

in points outside of matter. Here λ is a constant and $G_{\mu\nu}$ the contracted Riemann-Christoffel tensor, an expression in the $g_{\mu\nu}$ and their first and second derivatives.†

It is the purpose of this paper to restrict the functions $g_{\mu\nu}$ a priori, to solve the equations (0.1) under such restrictions, and to interpret the solutions thus found. In particular, we examine those manifolds which may be described in terms of orthogonal coördinates x, y, z, t such that the 3-spaces t=constant are conformal euclidean with a ratio of magnification which depends on t. The line element of such a manifold may be written

$$ds^2 = \frac{1}{\rho^2}(dx^2 + dy^2 + dz^2 + \sigma^2 dt^2) \tag{0.2}$$

where ρ and σ are functions of all four coördinates and where $\partial\rho(x,y,z,t)/\partial t \neq 0$. If we interpret t as the time coördinate of a physical space-time, then the velocity of light in this space is independent of its direction.‡ Physically stated, we propose to find all dynamical space-times which admit of representation by orthogonal coördinates in which the velocity of light is isotropic.

It is not to be expected that all of the solutions found will represent different manifolds, for any transformation under which the line element (0.2) is invariant will carry any one of our solutions into another (or into itself) and in this case the two solutions must be considered as defining

* Presented to the Society, San Francisco Section, June 19, 1925; received by the editors in December, 1925.

† The notation here used is that of Eddington's *The Mathematical Theory of Relativity*, 1923, in which a complete discussion of the theory may be found.

‡ Eddington, loc. cit., pp. 40, 93.

the same manifold. Of these transformations the most interesting from the physical standpoint are those which leave the time axis unaltered, thus allowing us to find to what extent the spacial properties are determined for a given observer in a manifold of this type. This requires that the conformal euclidean space t=constant remain conformal euclidean; it has been shown* that the most general transformation satisfying this requirement can be produced by successive applications of linear orthogonal transformations and inversions. We shall find it advantageous to consider as the elemental transformation of this latter group inversion with respect to a point on the circle at infinity, defined by

$$(0.3)\quad x = \frac{kx'}{y' + iz'}, \qquad y = \frac{r'^2 - k^2}{2(y' + iz')}, \qquad z = \frac{r'^2 + k^2}{2i(y' + iz')}, \qquad t = t'.\dagger$$

Under this transformation the line element (0.2) goes into

$$(0.31)\qquad ds^2 = \frac{1}{\rho'^2}(dx'^2 + dy'^2 + dz'^2 + \sigma'^2 dt'^2),$$

where

$$\rho'(x',y',z',t') = \frac{y' + iz'}{k}\rho(x,y,z,t), \qquad \sigma'(x',y',z',t') = \frac{y' + iz'}{k}\sigma(x,y,z,t),$$

in which x, y, z, t are the functions of x', y', z', t' defined by (0.3). Hence if we have a solution $\rho(x, y, z, t)$, $\sigma(x, y, z, t)$ the transformation yields another solution $\rho'(x', y', z', t')$, $\sigma'(x', y', z', t')$ which must already be contained among those found by direct methods; we are then justified in limiting ourselves to the discussion of those solutions which are not equivalent under transformations of this type.

1. **The equations.** The ten equations (0.1) may, in the case of a manifold given by (0.2), be replaced by the set

A $\qquad G_{14} = G_{24} = G_{34} = 0,$

B $\qquad G_{23} = G_{31} = G_{12} = 0,$

C $\qquad G_{11} = G_{22} = G_{33},$

D $\qquad G_{11} + G_{22} + G_{33} - (1/\sigma^2)G_{44} = (2/\rho^2)\lambda,$

E $\qquad G_{44} = (\sigma^2/\rho^2)\lambda.$

---

* Liouville, *Note au sujet de l'article précédent*, Journal de Mathématiques, vol. 12 (1847), p. 265.
† Darboux, *Systèmes Orthogonaux*, 1910, p. 168.

Expressing the $G_{\mu\nu}$ in terms of ρ and σ set A becomes

$$\rho_{14} - \rho_4 \frac{\sigma_1}{\sigma} = \rho_{24} - \rho_4 \frac{\sigma_2}{\sigma} = \rho_{34} - \rho_4 \frac{\sigma_3}{\sigma} = 0,$$

where the subscripts 1, 2, 3, 4 indicate differentiation with respect to x, y, z, t respectively. The solution of these equations is

$$\sigma = \frac{1}{f(t)} \rho_4,$$

where f is an arbitrary function of t. Using this value of σ equations B and C become

$$\text{B} \qquad \frac{\rho_{234}}{\rho_4} - \frac{2\rho_{23}}{\rho} = 0, \text{ etc.},$$

$$\text{C} \qquad \frac{\rho_{114}}{\rho_4} - \frac{2\rho_{11}}{\rho} = \frac{\rho_{224}}{\rho_4} - \frac{2\rho_{22}}{\rho} = \text{etc.},$$

whence on integration

$$\rho_{23} = \rho^2 a(x,y,z), \text{ etc.},$$

$$\rho_{22} - \rho_{33} = \rho^2 l(x,y,z), \text{ etc.},$$

in which $a, \cdots, l, \cdots$ are arbitrary functions of x, y and z. Equations D and E may also be expressed in terms of ρ and its derivatives; the latter is however not independent (D is a first integral of it) and may be discarded.

These results may be stated as follows: *A manifold whose metrical relations are determined by* (0.2) *is an Einstein field* (i. e., satisfies equations (0.1)) *provided*

$$(1.0)\quad \begin{array}{lll} \text{A} & \sigma = \dfrac{1}{f(t)}\rho_4, & \rho_4 \neq 0, \\ \text{B} & \rho_{23} = \rho^2 a(x,y,z), \quad \rho_{31} = \rho^2 b(x,y,z), & \rho_{12} = \rho^2 c(x,y,z), \\ \text{C} & \rho_{22} - \rho_{33} = \rho^2 l(x,y,z), \quad \rho_{33} - \rho_{11} = \rho^2 m(x,y,z), & \\ & \rho_{11} - \rho_{22} = \rho^2 n(x,y,z), & \\ \text{D} & -2\rho(\rho_{11} + \rho_{22} + \rho_{33}) + 3(\rho_1^2 + \rho_2^2 + \rho_3^2) = \lambda - 3f^2(t), & \end{array}$$

where a, b, c, l, m, n, and f are arbitrary functions of their arguments (except for the identity $l+m+n=0$). The problem thus reduces to that of the solution of six second-order non-linear partial differential equations in one dependent and three independent variables; we must now investigate under what conditions an integral of these equations may exist.

2. **The conditions of integrability.** We next develop certain necessary conditions of integrability for the equations (1.0) above.

First, since

$$\frac{\partial \rho_{23}}{\partial x} = \frac{\partial \rho_{31}}{\partial y} = \frac{\partial \rho_{12}}{\partial z},$$

set B requires that

$$\rho_1 a + \frac{1}{2}\rho a_1 = \rho_2 b + \frac{1}{2}\rho b_2 = \rho_3 c + \frac{1}{2}\rho c_3. \tag{2.1}$$

Next,

$$\frac{\partial}{\partial x}(\rho_{22} - \rho_{33}) = \frac{\partial \rho_{12}}{\partial y} - \frac{\partial \rho_{31}}{\partial z},$$

whence from B and C

$$\rho_1 l - \rho_2 c + \rho_3 b + \tfrac{1}{2}\rho(l_1 - c_2 + b_3) = 0. \tag{2.2}$$

To this we must add the two other conditions obtained by the cyclic permutation (123) (*abc*) (*lmn*).

The third set of conditions is derived from equation D in conjunction with B and C. With the aid of the second two equations of C, D may be written

$$-\rho(\rho_{22} + \rho_{33}) + \frac{1}{3}\rho^3(m - n) + \rho_1^2 + \rho_2^2 + \rho_3^2 = \frac{\lambda}{3} - f^2(t).$$

Differentiating with respect to x and simplifying the resulting equation by means of B we find

$$b_3 + c_2 - \tfrac{1}{3}(m_1 - n_1) = 0. \tag{2.3}$$

Permutation gives the two other equations of this type.

The conditions (2.1) and (2.2) are five linear homogeneous equations in the four quantities ρ_1, ρ_2, ρ_3 and ρ, which requires that a, l, etc., satisfy the two eliminants of the equations. We may then, in general, solve for the derivatives of ρ and demand that the values thus found are consistent with the original equations; we here develop these additional conditions for the general case $abc \neq 0$ and consider the alternative cases (which can be handled by simpler methods) as they arise.

In the case here considered ($abc \neq 0$) we may eliminate ρ_2 and ρ_3 from the first equation of (2.2) by means of (2.1) and obtain as eliminant

$$\rho_1 A_1' + \rho A_1 = 0 \tag{2.21}$$

where

$$A_1' = bcl - a(c^2 - b^2),$$

$$A_1 = \tfrac{1}{2}\left[bc(l_1 - c_2 + b_3) - a_1(c^2 - b^2) + b_2c^2 - c_3b^2\right].$$

The two remaining equations of (2.2) give similar conditions on ρ_2 and ρ_3, in which the coefficients are the cyclic permutations of the above. These equations are equivalent to (2.2) in the case here considered.

In order that (2.1) be consistent with B we must have

$$\frac{\partial\rho_2}{\partial z} = \frac{\partial}{\partial z}\left(\rho_1\frac{a}{b} + \rho\frac{a_1 - b_2}{2b}\right) = \rho^2 a.$$

On eliminating ρ_{13} and ρ_3 this becomes

$$\rho_1 B_1' + \rho B_1 = 0, \tag{2.4}$$

where

$$B_1' = a\left(\frac{a_3}{a} - \frac{b_3}{b} + \frac{a_1 - b_2}{2c}\right),$$

$$B_1 = \frac{1}{2}\left[(a_1 - b_2)_3 + (a_1 - b_2)\left(\frac{a_1 - c_3}{2c} - \frac{b_3}{b}\right)\right].$$

Similarly, in order that (2.1) be consistent with C

$$\frac{\partial\rho_2}{\partial y} - \frac{\partial\rho_3}{\partial z} = \frac{\partial}{\partial y}\left(\rho_1\frac{a}{b} + \rho\frac{a_1 - b_2}{2b}\right) - \frac{\partial}{\partial z}\left(\rho_1\frac{a}{c} + \rho\frac{a_1 - c_3}{2c}\right) = \rho^2 l,$$

and on simplifying as in the previous equation we find

$$\rho_1 C_1' + \rho C_1 = \rho^2\frac{A_1'}{bc}, \tag{2.5}$$

where A_1' is known from (2.21) above and

$$C_1' = \frac{a_2}{b} - \frac{a_3}{c} + \frac{a}{2b^2}(a_1 - 3b_2) - \frac{a}{2c^2}(a_1 - 3c_3),$$

$$C_1 = \frac{1}{2}\left[\frac{1}{b}(a_1 - b_2)_2 + \frac{1}{2b^2}(a_1 - b_2)(a_1 - 3b_2) - \frac{1}{c}(a_1 - c_3)_3 - \frac{1}{2c^2}(a_1 - c_3)(a_1 - 3c_3)\right].$$

As before, to the equations (2.4) and (2.5) are to be added those obtained by the cyclic permutation (123) (*abc*) (*lmn*).

The 14 equations which have been derived in this section are not independent, but we have written them in this form for the sake of symmetry.

3. **Solution of the conditions of integrability.** The conditions obtained in the preceding section may be considered as differential equations with dependent variables ρ, a, l, etc., which must be solved before we can integrate the original set (1.0). In order to accomplish this it is only necessary to consider two cases: (1), in which the three logarithmic derivatives ρ_1/ρ, ρ_2/ρ, and ρ_3/ρ of ρ do not depend on t, and (2) in which all three of them do. The only alternatives (cases in which one or two only of these three quantities are functions of t) can be reduced to (2) by rotating the x, y, z coördinate system.

In case (1), ρ must be of the form

$$\rho = \kappa(t)\tau(x,y,z)$$

in order that the three logarithmic derivatives be independent of t. But the first equation of (1.0B) requires that

$$\tau_{23} = \tau^2 a\kappa(t)$$

and since a and τ cannot depend on t this is only possible if $a=0$. Applying this reasoning to the remaining equations (1.0B) and (1.0C) it is seen that b, c, l, m, and n must also vanish, so we have

$$a = b = c = l = m = n = 0 \tag{3.1}$$

as the solution of the conditions of integrability for this case.

In order to deal with the conditions of integrability for case (2) in the form in which they were developed in § 2 above we first dispose of the possibility $abc=0$. Then one, and consequently all, of the three quantities a, b and c must vanish; otherwise it would be possible to solve (2.1) for at least one of the logarithmic derivatives of ρ in terms of x, y and z alone, which is contradictory to the assumption that they all depend on t. Equations (2.2) then require, for the same reason, that l, m and n also vanish; we thus arrive at the solution (3.1) above.

We may now take as our equations the sets (2.1)-(2.5). Since a, l, etc., do not depend on t, (2.21) and (2.4) can only be satisfied if all coefficients A_1, A_1', B_1, B_1' and their cyclic permutations vanish. But then the right hand side of (2.5) is zero, so we must also have $C_1=C_1'=0$, etc. The vanishing of coefficients A_i' enables us to express l, m and n in terms of the variables a, b and c:

$$l = \frac{a}{bc}(c^2 - b^2), \quad m = \frac{b}{ca}(a^2 - c^2), \quad n = \frac{c}{ab}(b^2 - a^2). \tag{3.2}$$

The conditions of integrability for this case are thus reduced to the 15 equations arising from the vanishing of the coefficients mentioned above, the set (2.3) and the set (2.1). The first 18 of these are, on eliminating l, m and n by (3.2), differential equations with dependent variables a, b and c which are now to be integrated.

12 of these equations (those arising from A_i, B_i', and C_i', and (2.3)) are of first order, but it is found that of these but 6 are linearly independent. As independent equations we may take two of the set $A_i=0$, all of $B_i'=0$, and one of (2.3):

$$
(3.3)\qquad
\begin{aligned}
&\frac{b_1}{b}+\frac{b_2}{2a}=\frac{c_1}{c}+\frac{c_3}{2a},\\
&\frac{c_2}{c}+\frac{c_3}{2b}=\frac{a_2}{a}+\frac{a_1}{2b},\\
&\frac{a_3}{a}+\frac{a_1}{2c}=\frac{b_3}{b}+\frac{b_2}{2c},\\
&-a_1(c^2-b^2)+b_2c^2-c_3b^2+bc(l_1-c_2+b_3)=0,\\
&-b_2(a^2-c^2)+c_3a^2-a_1c^2+ca(m_2-a_3+c_1)=0,\\
&c_2+b_3=\tfrac{1}{3}(m_1-n_1),
\end{aligned}
$$

where l, m, and n are given by (3.2). It can be shown that the conditions of complete integrability are not satisfied by this set.*

In order to integrate the second order equations we first note that (3.3) may, in general, be solved for all other derivatives in terms of those with respect to any one, say z.† This being done, it can be shown that

$$
(3.3')\qquad \frac{a_1-c_3}{2c}=-\frac{1}{3}\frac{\partial}{\partial z}\log\left(a^4bc\Delta^{5/2}\right),
$$

where

$$
\Delta=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\cdot
$$

With the aid of this relation the equation $B_1=0$((2.4) above) becomes

$$
(a_1-b_2)_3-\frac{1}{3}(a_1-b_2)\frac{\partial}{\partial z}\log\left(a^4b^4c\Delta^{5/2}\right)=0,
$$

* Forsyth, *Theory of Differential Equations*, vol. V, 1902, chap. XI.

† The determinant involved is a multiple of Δ, and the case in which it vanishes is discussed later.

whence

$$(3.31) \qquad a_1 - b_2 = (a^4b^4c\Delta^{5/2})^{1/3}\gamma(x, y).$$

Similarly

$$b_2 - c_3 = (ab^4c^4\Delta^{5/2})^{1/3}\alpha(y, z),$$

$$c_3 - a_1 = (a^4bc^4\Delta^{5/2})^{1/3}\beta(z, x).$$

On putting the last of these back into (3.3′) we find, introducing the notation $au = bv = cw = 2(abc\Delta^{5/2})^{-1/3}$ $(= 8uvw(u^2+v^2+w^2)^{-5/2})$,

$$(3.32) \qquad \frac{\partial v}{\partial z} = -\beta(z, x),$$

whence by symmetry considerations

$$\frac{\partial u}{\partial y} = \gamma(x, y).$$

Four additional equations of this type are obtained by the cyclic permutation (xyz) (uvw) $(\alpha\beta\gamma)$.

The equations $C_i = 0$ now offer a means of determining the so far arbitrary functions α, β and γ; $C_1 = 0$ may be written, after manipulation analogous to that applied above,

$$\frac{1}{b}(a_1 - b_2)\frac{\partial}{\partial y}\log\left[(a_1 - b_2)(a^4b^4c\Delta^{5/2})^{-1/3}\right]$$

$$-\frac{1}{c}(a_1 - c_3)\frac{\partial}{\partial z}\log\left[(a_1 - c_3)(a^4bc^4\Delta^{5/2})^{-1/3}\right] = 0,$$

and on introducing (3.31) this becomes

$$\frac{\partial\gamma(x, y)}{\partial y} + \frac{\partial\beta(z, x)}{\partial z} = 0.$$

The remaining equations $C_2 = C_3 = 0$ yield

$$\frac{\partial\alpha(y, z)}{\partial z} + \frac{\partial\gamma(x, y)}{\partial x} = 0, \qquad \frac{\partial\beta(z, x)}{\partial x} + \frac{\partial\alpha(y, z)}{\partial y} = 0.$$

The solution of these three equations is readily found to be

$$\alpha = Cy - Bz + A', \qquad \beta = Az - Cx + B', \qquad \gamma = Bx - Ay + C',$$

where A, A', etc., are constants.

(3.32) may now be integrated, giving

$$u = -\tfrac{1}{2}Ar^2 + x(Ax + By + Cz) + C'y - B'z + X(x),$$

and two similar equations for v and w. Equations (3.31) then become, on using the expressions for a, b, and c in terms of u, v, and w,

$$u_1 = v_2 = w_3, \qquad \alpha u + \beta v + \gamma w = 0.$$

The first two of these require that $X'(x) = Y'(y) = Z'(z)$, i. e.

$$X = -\epsilon x + a', \qquad Y = -\epsilon y + b', \qquad Z = -\epsilon z + c',$$

where ϵ, a', b' and c' are constants; the last relation imposes the conditions

$$\epsilon A' = Bc' - Cb', \qquad \epsilon B' = Ca' - Ac', \qquad \epsilon C' = Ab' - Ba'$$

on the constants involved, and if $\epsilon = 0$ we must add to these the condition $AA' + BB' + CC' = 0$.

The values of a, b, and c are now given by

$$a = \frac{K}{u}, \qquad b = \frac{K}{v}, \qquad c = \frac{K}{w}, \tag{3.4}$$

where

$$K = 2(abc\Delta^{5/2})^{-1/3} = 8uvw(u^2 + v^2 + w^2)^{-5/2},$$
$$u = -\tfrac{1}{2}Ar^2 + x(Ax + By + Cz - \epsilon) + C'y - B'z + a', \text{ etc.},$$

and $\epsilon A' = Bc' - Cb'$, etc.; $AA' + BB' + CC' = 0$. These values satisfy the first order equations (3.3) as well as those of second order, and represent the general solution of the conditions of integrability insofar as they do not involve ρ (except for the case $\Delta = 0$). This solution may be considerably simplified by translation of the x, y, z coördinate system as follows:

(i) The general case in which not all three of the constants A, B and C vanish becomes, on referring it to a coördinate system in which A', B' and C' vanish,*

$$u = -\tfrac{1}{2}A(r^2 - \delta^2) + x(Ax + By + Cz - \epsilon), \text{ etc.} \tag{3.41}$$

(ii) If $A = B = C = 0$ we may write, on translation,

$$u = C'y - B'z, \text{ etc.}, \tag{3.42}$$

provided not all A', B' and C' vanish.

---

* If, for example, $A \neq 0$, this may be accomplished by transferring the origin to the point $(0, -C'/A, B'/A)$. Further simplification could be accomplished by a rotation, but the symmetry would be destroyed.

(iii) The only remaining case, that in which all the A, A', etc., vanish, is given by

$$u = -\epsilon x + a', \text{ etc.} \tag{3.43}$$

In order to determine ρ we must now return to the set (2.1) and substitute in it the known values of a, b and c. It then becomes

$$w\rho_2 - v\rho_3 + \alpha\rho = 0, \quad v\rho_3 - w\rho_1 + \beta\rho = 0, \quad v\rho_1 - u\rho_2 + \gamma\rho = 0, \tag{3.45}$$

of which but two equations are independent. On eliminating ρ between two of these we find

$$\alpha\rho_1 + \beta\rho_2 + \gamma\rho_3 = 0, \tag{3.45'}$$

which may be used in place of one of (3.45) (unless α, β and γ all vanish). The three cases into which the general solution is divided are the following:

(i) Three independent integrals of the characteristic equations of (3.45′)

$$\frac{dx}{Cy - Bz} = \frac{dy}{Az - Cx} = \frac{dz}{Bx - Ay} = \frac{d\rho}{0}$$

are $\rho = \text{constant}$, $Ax+By+Cz = \text{constant}$ and $x^2+y^2+z^2=r^2=\text{constant}$. Hence $\rho=\rho(Ax+By+Cz,\ r,\ t)$, and (3.45) becomes

$$(\xi - \epsilon)\rho_\xi + (\eta - \delta^2)\rho_\eta = \rho,$$

where ξ and η are the two first arguments of ρ. The functional form of ρ is consequently given for this case by

$$\rho = (Ax + By + Cz - \epsilon)\tau\left(\frac{r^2 - \delta^2}{Ax + By + Cz - \epsilon}, t\right). \tag{3.51}$$

Not all three $A^2+B^2+C^2$, δ^2 and ϵ may vanish simultaneously, for we would then have $\Delta=0$.

(ii) The values of a, b and c given by (3.42) lead, by the same method, to the solution

$$\rho = (C'x - A'z)\tau\left(\frac{A'y - B'z}{C'x - A'z}, t\right). \tag{3.52}$$

(iii) Corresponding to the third case, the equations (3.45) become

$$\frac{\rho_1}{\epsilon x - a'} = \frac{\rho_2}{\epsilon y - b'} = \frac{\rho_3}{\epsilon z - c'}.$$

If $\epsilon \neq 0$ these yield

$$\rho = \rho(r, t). \tag{3.53}$$

In case $\epsilon=0$ the solution is

$$\rho = \rho(Ax + By + Cz, t), \tag{3.54}$$

where we have written A, B, and C in place of a', b' and c'; these constants are, for the present, subject to the restriction $A^2+B^2+C^2\neq 0$.

The only case remaining to be examined is that in which $\Delta=0$. It can be shown, by a rather tedious analysis which is omitted because of the triviality of the results, that the only solutions arising from this case may be included in (3.51) and (3.54) by removing the restrictions there imposed on the constants $A^2+B^2+C^2$, δ^2 and ϵ. Only the ratios $a:b:c$ are given by (3.33) for these cases.

4. **Solution of the equations.** We have thus solved the conditions of integrability, and may proceed to the solution of the original equations. Since, however, of the five solutions obtained ((3.1) and (3.51)-(3.54)) the last four are characterized by the fact that in them the functional dependence of ρ is already given, we first examine them in accordance with the remarks made in the introduction to see whether they can lead to distinct manifolds. On subjecting these four solutions to transformations of the form (0.3) there result solutions (0.31) which are already among the four; in particular, it can be shown that (3.52)-(3.54) can be obtained from (3.51) (or special cases thereof), and that (3.53) and (3.54) are together fully equivalent to it. It is therefore only necessary to consider solutions of (1.0) arising from (3.1) and (3.51) or (and this is the course we shall follow) those arising from (3.1), (3.53) and (3.54).

In the first case to be considered the six quantities $a \cdots l \cdots$ vanish, so the first five equations of (1.0) are

$$\rho_{23} = \rho_{31} = \rho_{12} = 0, \qquad \rho_{11} = \rho_{22} = \rho_{33}.$$

The solution of these is

$$\rho = d(t)(x^2 + y^2 + z^2) + d_1(t)x + d_2(t)y + d_3(t)z + d_4(t), \tag{4.1}$$

where the d's are arbitrary functions. (1.0D) requires that

$$f^2(t) = \frac{\lambda}{3} - d_1^2 - d_2^2 - d_3^2 + 4dd_4,$$

and on introducing these values of ρ and $f(t)$ into (1.0A) we obtain σ. Substituting ρ and σ into the components of the Riemann-Christoffel tensor $B_{\alpha\beta\gamma\delta}$* we find that the manifold defined by this case is a hypersphere of

* Eddington, loc. cit., p. 72.

Riemann curvature $-\lambda/3$. This solution is equivalent to one obtained by E. Kasner.*

It is readily shown that the only manifolds of type (0.2) in which the coefficient of dt^2 is a function of t alone are special cases of (4.1); we may deduce from this the impossibility of stationary observers (as their existence implies that the world is not "empty") employing orthogonal coordinates, of which their proper time is one, in which the velocity of light is isotropic, unless their fundamental intervals are functions of their position.

For the second case, (3.53), in which ρ is a function of r and t, the equations (1.0) are

$$\frac{\partial^2\rho}{\partial r^2} - \frac{1}{r}\frac{\partial\rho}{\partial r} = \left(\frac{2}{\epsilon}\right)^3 \frac{\rho^2}{r^3},$$

$$-\frac{2}{3}\rho\frac{\partial^2\rho}{\partial r^2} - \frac{4}{3}\frac{\rho}{r}\frac{\partial\rho}{\partial r} + \left(\frac{\partial\rho}{\partial r}\right)^2 = \frac{\lambda}{3} - f^2(t).$$

On eliminating the second derivative of ρ these lead to the equivalent first order equation

$$\left(\frac{\partial\rho}{\partial r}\right)^2 - 2\frac{\rho}{r}\frac{\partial\rho}{\partial r} - 4C\frac{\rho^3}{r^3} = \frac{\lambda}{3} - f^2(t)$$

where $C = (4/3)\ (1/\epsilon^3)$. The solution of this is

$$\rho = \frac{r}{C}\left\{\wp(\log rg(t)) - \frac{1}{12}\right\}, \tag{4.2}$$

in which $\wp$ is a Weierstrass $\wp$-function whose invariants are

$$g_2 = \frac{1}{12}, \quad g_3 = -\frac{1}{216} - \left(\frac{\lambda}{3} - f^2(t)\right)C^2.$$

In this form the line element contains elliptic functions, and it is difficult to reconcile their doubly-periodic property with any attempted physical interpretation;† as will be shown below, however, this behavior

* E. Kasner, American Journal of Mathematics, vol. 43 (1921), pp. 20–28; Mathematische Annalen, vol. 85 (1922), pp. 234–236.

† The difficulty of this reconciliation can be seen from the fact that although $\wp$ need not be periodic in log r, it must at least be possible to find at any time t values for which $|\wp|$ assumes any assigned value. Two integrals of the equations of motion of a particle placed in this field are

$$\theta = \frac{\pi}{2}, \qquad \left(\frac{r}{\rho}\right)^2\frac{d\varphi}{ds} = \left(\frac{C}{\wp}\right)^2\frac{d\varphi}{ds} = \text{const.},$$

and these, because of the above mentioned property of $\wp$, would require a rather erratic behavior. (Cf. Eddington, loc. cit., pp. 60, 85.)

is not inherent in the manifold, but is a property of the coördinate system we have chosen. In order to avoid this objection we require that $\wp$ degenerate to a singly periodic function in such a way that it is not periodic along the real axis; this is only possible if we take the root $f^2(t)=\lambda/3$ of the condition $g_2^3-27g_3^2=0$. $\wp$ is then, on integration,

$$\wp = \tfrac{1}{4}\tanh^2\log\,(rg(t))^{1/2} - \tfrac{1}{6},$$

and on simplifying and substituting in (4.2),

$$\rho = -\frac{1}{Cg(t)}\left(1+\frac{1}{rg(t)}\right)^{-2};$$

t only appears in ρ in the parameter $g(t)$, so the coefficient of dt^2 in the line element contains $[g'(t)]^2$ as a factor; hence no loss of generality is incurred by choosing g in any way we please. In particular, we write $C=im/2$ and choose $g(t)=(2/m)e^{\kappa t}$ where $\kappa^2=\lambda/3$. Then

$$\text{(4.21)} \qquad ds^2 = -e^{-2\kappa t}\left(e^{\kappa t}+\frac{m}{2r}\right)^4(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2) + \frac{\left(e^{\kappa t}-\frac{m}{2r}\right)^2}{\left(e^{\kappa t}+\frac{m}{2r}\right)^2}dt^2.$$

The manifold defined by (4.21) reduces to a special case of (4.1), and therefore represents a hypersphere, if $m=0$. If $\kappa=0$ the coefficients, although no longer containing t, still satisfy the equations (0.1); the line element is, in fact, the isotropic form of the Schwarzschild element for that case.* On the other hand, it is well known that the general Schwarzschild solution

$$\text{(4.2')} \qquad \begin{aligned} ds^2 &= -\left(\frac{dR^2}{\gamma}+R^2d\Theta^2+R^2\sin^2\Theta d\Phi^2\right)+\gamma dT^2,\\ \gamma &= 1-\frac{2m}{R}-\kappa^2R^2 \end{aligned}$$

represents the same fields in these two limiting cases; it therefore seems reasonable to ask if (4.21), or the apparently more general (4.2) from which it was derived, is not equivalent to the Schwarzschild element (4.2′). That

* For this and the following, see Eddington, loc. cit., pp. 93, 100.

this is, in fact, the case can be shown by subjecting (4.2′) to the transformation

$$R = R(r,t), \quad \Theta = \theta, \quad \Phi = \phi, \quad T = T(r,t),$$

and requiring that the resulting expression be of the form (0.2). (4.2′) then becomes (the subscripts r and t denoting differentiation)

$$ds^2 = -\frac{R^2}{r^2}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) + \left(\gamma T_t^2 - \frac{1}{\gamma} R_t^2\right) dt^2,$$

provided R and T satisfy

$$\gamma T_r T_t = \frac{1}{\gamma} R_r R_t, \qquad \frac{R_r^2}{\gamma} - \gamma T_r^2 = \frac{R^2}{r^2}\cdot$$

On solving these two equations for T_r and T_t and applying the condition of integrability

$$\frac{\partial T_r}{\partial t} = \frac{\partial T_t}{\partial r}$$

we obtain

$$r^2 R R_{rr} - 2r^2 R_r^2 + rRR_r - \frac{1}{2}\gamma' R^3 + \gamma R^2 = 0, \qquad \gamma' = \frac{d\gamma}{dR},$$

whence, on multiplying by $2R_r/R^5$ and integrating with respect to r,

$$\frac{r^2 R_r^2}{R^4} - \frac{\gamma}{R^2} = f^2(t),$$

a function of t alone. Changing from dependent variable R to $\rho = ir/R$ this equation becomes

$$\left(\frac{\partial \rho}{\partial r}\right)^2 - 2\frac{\rho}{r}\frac{\partial \rho}{\partial r} - 2mi\left(\frac{\rho}{r}\right)^3 = \frac{\lambda}{3} - f^2(t),$$

and the coefficient of dt^2

$$\left(\frac{\partial \rho}{\partial t}\right)^2 \frac{1}{\rho^2 f^2(t)}\cdot$$

But the line element defined in this case is seen to be identical with (4.2) on setting $m = -2iC$. *Hence our "dynamical" spherically symmetric solution is equivalent to the Schwarzschild statical solution*!

The line element (4.21) is equivalent to the one defined by (4.2), from which it was obtained by setting $f^2(t)=\lambda/3$, since the latter is equivalent to (4.2′) for *all* values of $f(t)$. It can be obtained directly from the Schwarzschild form by the transformation

$$R = re^{-\kappa t}\left(e^{\kappa t} + \frac{m}{2r}\right)^2, \qquad \Theta = \theta, \qquad \Phi = \phi,$$

$$T = t + 8m^2\kappa F\left(\frac{e^{\kappa t} - m/(2r)}{e^{\kappa t} + m/(2r)}\right),$$

where

$$F(\xi) = \int \frac{d\xi}{(1-\xi^2)[\xi^2(1-\xi^2)^2 - 4m^2\kappa^2]}\,.$$

The evaluation of $F(\xi)$ involves the solution of the cubic equation $\xi^3-\xi\pm 2m\kappa=0$, and is not necessary for our purpose.

The last case, $\rho=\rho(Ax+By+Cz,\ t)$, leads to two solutions, according as $A^2+B^2+C^2$ vanishes or not. In the latter case we may, on rotation of the coördinate system, bring ρ into the form $\rho=\rho(x,\ t)$. Equation (1.0) then becomes

$$\left(\frac{\partial \rho}{\partial x}\right)^2 = 4C\rho^3 + \frac{\lambda}{3} - f^2(t)$$

of which the solution is

$$\rho = \wp(C^{1/2}x + g(t)), \qquad g_2 = 0, \qquad g_3 = -\frac{1}{C}\left(\frac{\lambda}{3} - f^2(t)\right). \tag{4.3}$$

Using the methods employed in the preceding case, it can be shown that this solution is equivalent to

$$ds^2 = \frac{4dX^2}{K^2(1+X^2)^2} + \left(\frac{2}{1+x^2}\right)^{2/3}(dY^2+dZ^2+X^2dT^2), \qquad K^2 = -9\kappa^2,$$

which is a special case of a solution obtained by E. Kasner.* As before, we may choose $f^2(t)=\lambda/3$, and obtain the equivalent solution

$$ds^2 = (x-\kappa t)^4(dx^2+dy^2+dz^2) + 4(x-\kappa t)^{-2}dt^2. \tag{4.31}$$

The case $\kappa=0$ of this solution has been given explicitly by Kasner.

* E. Kasner, these Transactions, vol. 27 (1925), p. 155.

If $A^2+B^2+C^2=0$, we may take ρ in the form $\rho=\rho(x+iy, t)$. Then (1.0) becomes

$$\frac{\partial^2\rho}{\partial\xi^2} = \phi(\xi)\rho^2, \qquad \xi = x + iy, \tag{4.4}$$

where ϕ is arbitrary and $f^2(t)=\lambda/3$. It is not possible to give this solution a physical interpretation if we consider x, y and z as the space coördinates, for the coefficients of the line element would then be functions of a complex variable. In order to avoid this we could interchange the rôle of x or y with t, but the velocity of light would no longer be isotropic.

In conclusion, we have found that there exist four distinct Einstein fields whose line elements are of the form (0.2), given by (4.1), (4.21), (4.31) and (4.4). These represent (1) a hypersphere, (2) a dynamical form of the statical Schwarzschild solution, (3) a dynamical form of a solution given by Kasner, a "one-dimensional state of motion," and (4) a rather general class given by the solution of any equation of the type (4.4), which involves one essential arbitrary function of t. The question of whether we may consider the r and t of (2) as physically observed quantities has not been here considered, nor have we attempted to determine whether all statical solutions of the form (0.2) may on transformation be thrown into the dynamical form, as was done in this case.

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REDUCTION OF THE ORDINARY LINEAR DIFFERENTIAL EQUATION OF THE nTH ORDER WHOSE COEFFICIENTS ARE CERTAIN POLYNOMIALS IN A PARAMETER TO A SYSTEM OF n FIRST-ORDER EQUATIONS WHICH ARE LINEAR IN THE PARAMETER*

BY

CHARLES E. WILDER

The ordinary linear differential equation of the nth order herein considered,

$$\sum_{k=0}^{n} \frac{d^k y}{dx^k} P_k(x,\rho) = 0, \tag{1}$$

in which

$$\begin{aligned} P_k(x,\rho) &= \sum_{j=0}^{n-k} P_{n-k,j}(x)\rho^j, \quad k \neq n, \\ &= 1, \quad k = n, \end{aligned} \tag{2}$$

is a special case of the equation for which Birkhoff developed asymptotic solutions,† but more general than the one for which he solves the expansion problem.‡ In the form given above the equation with various types of boundary conditions has been extensively studied by Tamarkin.§ On the other hand Birkhoff and Langer|| have treated the boundary problem and developments associated with the system of ordinary linear differential equations of the first order

$$\frac{dy_i}{dx} = \sum_{j=1}^{n} A_{ij}(x,\rho) y_j \qquad (i = 1, 2, \cdots, n), \tag{3}$$

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* Presented to the Society, September 9, 1926; received by the editors in April, 1926.

† Birkhoff, *On the asymptotic character of the solutions of certain linear differential equations containing a parameter*, these Transactions, vol. 9 (1908), pp. 219–231.

‡ Birkhoff, *Boundary value and expansion problems of ordinary linear differential equations*, these Transactions, vol. 9 (1908), pp. 373–395.

§ Tamarkin, *On certain general problems of the theory of ordinary linear differential equations and the expansion of an arbitrary function in series*, Petrograd, 1917 (Russian). Cf. also a paper under the same title which will appear shortly in the Mathematische Zeitschrift.

|| Birkhoff and Langer, *The boundary problems and developments associated with a system of ordinary linear differential equations of the first order*, Proceedings of the American Academy of Arts and Sciences, vol. 58 (1923), pp. 51–128.

in which

$$A_{ij}(x,\rho)=\rho a_{ij}(x)+b_{ij}(x) \qquad (i,j=1,2,\cdots,n). \tag{4}$$

It is the object of this paper to study the relation between the equation (1) and the system (3), and in particular to prove the following theorem:*

If the equation (1) *is such that the functions* $P_{i+k,i}(x)$ *possess* $n-k-1$ *continuous derivatives in the interval* $a\leqq x\leqq b$, *for* $i=1, 2, \cdots, n$, *and if the "characteristic equation"*

$$\sum_{t=0}^{n} P_{n-t,n-t}(x)a^t = 0 \tag{5}$$

has roots which are distinct for all values of x *in the interval, then the equation* (1) *may be reduced to a system of the form* (3), *in which the functions* $a_{ij}(x)$ *and* $b_{ij}(x)$ *are continuous in the same interval.*

In the system (3) set

$$y = \sum_{j=1}^{n} C_{0j}y_j. \tag{6}$$

From the equations obtained by differentiation,

$$y^{(k)} = \sum_{j=1}^{n} C_{kj}y_j \qquad (k=1,2,\cdots,n), \tag{7}$$

in which

$$C_{kj} = \sum_{r=1}^{n} C_{k-1,r}A_{rj} + C'_{k-1,j} \qquad (j,k=1,2,\cdots,n), \tag{8}$$

the y_i may be eliminated and the nth order equation

$$\begin{vmatrix} y & C_{01} & C_{02} & C_{03} & \cdots & C_{0n} \\ y' & C_{11} & C_{12} & C_{13} & \cdots & C_{1n} \\ y'' & C_{21} & C_{22} & C_{23} & \cdots & C_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ y^{(n)} & C_{n1} & C_{n2} & C_{n3} & \cdots & C_{nn} \end{vmatrix} = 0 \tag{9}$$

* The simple substitution $y=y_1$, $y_i'=\rho y_{i+1}$, will reduce the equation (1) to a system of first-order equations, but they will not all be of the form in the parameter required in (3).

obtained. If then the coefficient of $y^{(k)}$ is denoted by P_k, there are between the P's and the C's the relations

$$\sum_{k=0}^{n} P_k C_{kj} \equiv 0 \qquad (j = 1, 2, \cdots, n). \tag{10}$$

It is now assumed that in the set (3)

$$\begin{aligned} a_{ij} &= 0, \qquad j > i, \\ b_{i,i+1} &= 1, \end{aligned} \tag{11}$$

and that all the other b_{ij} are zero except when $i = n$. It is to a system (3) of this sort that the equation (1) is reducible. From (11) follows immediately

$$\begin{aligned} A_{ij} &= 0, \quad j > i + 1, \\ &= 1, \quad j = i + 1, \\ &= \rho a_{ij}, \quad j < i + 1,\ i \neq n, \\ &= \rho a_{nj} + b_{nj}, \quad i = n. \end{aligned} \tag{12}$$

Moreover in the substitution (6) it is now assumed that

$$\begin{aligned} C_{0j} &= 1, \quad j = 1, \\ &= 0, \quad j \neq 1. \end{aligned}$$

And it follows that

$$C_{kj} = \begin{cases} 0, & j > k + 1, \\ 1, & j = k + 1. \end{cases} \tag{13}$$

This is obviously true for $k = 0$, while the recursion formula (8), modified by the restrictions (12) on the A_{ij}, furnishes the material for an easy proof by induction that it holds for any k.

In the expansion of (9) the coefficient of $y^{(n)}$ is unity because of (13), and a slight examination of the form of the C's in ρ, which will incidentally appear in what follows, shows that the equation (9) now actually has the form (1). That is, the system (3) if subject to the restrictions (11) can be transformed to an nth order equation of the form (1).

Under the restrictions (12) and (13) the recursion formula (8) becomes

$$\begin{aligned} C_{kj} &= C_{k-1,j-1} + \rho \sum_{r=j}^{k} C_{k-1,r} a_{rj} + C'_{k-1,j}, \quad k \neq n, \quad j \leqq k, \\ &= C_{n-1,j-1} + \rho \sum_{r=j}^{n} C_{n-1,r} a_{rj} + C'_{n-1,j} + b_{nj}, \quad k = n, \end{aligned}$$

in which of course a C with second subscript zero is identically zero. This is now a formula for $C_{kj}-C_{k-1,\,j-1}$. It can then be rewritten for $C_{k-1,j-1}-C_{k-2,j-2}$ and so on for diminishing values of the subscripts until the second subscript in the second C becomes zero. These formulas are then summed to obtain

$$C_{kj} = \rho \sum_{s=1}^{j} \sum_{r=s}^{k-j+s} C_{k-j+s-1,r}a_{rs} + \sum_{s=1}^{j} C'_{k-j+s-1,s}, \quad k \neq n,$$

in which for $k=n$ the term b_{nj} must be added. This is more convenient in a slightly altered form. Set $k=j+i$ and it becomes

$$(14) \qquad C_{j+i,j} = \rho \sum_{s=1}^{j} \sum_{r=s}^{s+i} C_{s+i-1,r}a_{rs} + \sum_{s=1}^{j} C'_{s+i-1,s}$$

$$(j + i \neq n\,;\ i = 0,1, \cdots ,n - j - 1),$$

in which for $j+i=n$ the term b_{nj} must be added.

In particular for $i=0$, since by (13) $C_{s-1,s}=1, C'_{s-1,s}=0$, this gives

$$(15) \qquad \begin{aligned} C_{jj} &= \rho \sum_{s=1}^{j} a_{ss}, \qquad & j \neq n,\\ &= \rho \sum_{s=1}^{n} a_{ss} + b_{nn}, \quad & j = n. \end{aligned}$$

Now $C_{j+i,j}$ is of degree $i+1$ in ρ. This is obviously true for $i=0$, and if it is assumed for $i=k$ it then follows at once from (14) for $i=k+1$. It is also clear that the term independent of ρ is missing except for $j+i=n$. Hence set

$$(16) \qquad \begin{aligned} C_{j+i,j} &= \sum_{k=1}^{i+1} E_{j,i,k}\rho^k, \qquad & i + j \neq n,\\ &= \sum_{k=1}^{i+1} E_{j,i,k}\rho^k + b_{nj}, \quad & i + j = n, \end{aligned}$$

and substitute in (14):

$$\begin{aligned} \sum_{k=1}^{i+1} E_{j,i,k}\rho^k &= \rho \sum_{s=1}^{j} \sum_{r=s}^{s+i-1} \sum_{k=1}^{s+i-1} E_{r,s+i-1-r,k}\rho^k a_{rs}\\ &\quad + \sum_{s=1}^{j} \sum_{k=1}^{i} E'_{s,i-1,k}\,\rho^k + \rho \sum_{s=1}^{j} a_{s+i,s}\\ &= \rho \sum_{s=1}^{j} \sum_{k=1}^{i} \sum_{r=s}^{s+i-k} E_{r,s+i-1-r,k}a_{rs}\rho^k\\ &\quad + \sum_{s=1}^{j} \sum_{k=1}^{i} E'_{s,i-1,k}\rho^k + \rho \sum_{s=1}^{j} a_{s+i,s}, \end{aligned}$$

so that the recursion formulas for the E's are

$$E_{j,i,k} = \sum_{s=1}^{j} \left\{ \sum_{r=s}^{s+i-k+1} E_{r,s+i-1-r,k-1} a_{rs} + E'_{s,i-1,k} \right\}, \quad k \neq 1, \tag{17}$$

in which $E_{s,i,i+2}=0$,

$$E_{j,i,1} = \sum_{s=1}^{j} \left\{ E'_{s,i-1,1} + a_{s+i,s} \right\}.$$

In particular for $k=i+1$ these become

$$E_{j,i,i+1} = \sum_{s=1}^{j} \left\{ E_{s,i-1,i} a_{ss} + E'_{s,i-1,i+1} \right\},$$

but since in an E the last subscript can be at most one greater than the second, the last term in the above is zero, so that

$$E_{j,i,i+1} = \sum_{s=1}^{j} E_{s,i-1,i} a_{ss}. \tag{17a}$$

But this may be rewritten with the first subscript on the E equal to $j-1$ instead of j, and the result combined with the above gives

$$E_{j,i,i+1} = E_{j-1,i,i+1} + a_{jj} E_{j,i-1,i}. \tag{18}$$

This again may be rewritten for descending values of i, and from the system of equations all E's with first subscript j and second subscript less than i may be eliminated. Thus is obtained the formula

$$E_{j,i,i+1} = \sum_{k=0}^{i+1} a_{jj}^{k} E_{j-1,i-k,i-k+1} \tag{19}$$

in which $E_{j,-1,0}=1$.

It is now convenient to deduce an identity between these E's which will be of use in what follows. From (19)

$$E_{j,i-1,i} = \sum_{k=0}^{i} a_{jj}^{k} E_{j-1,i-k-1,i-k}.$$

Set $k+1=l$,

$$E_{j,i-1,i} = \sum_{l=1}^{i+1} a_{jj}^{l-1} E_{j-1,i-l,i-l+1},$$

and then multiply this by a_{jj}, obtaining

$$a_{jj} E_{j,i-1,i} - \sum_{l=1}^{i+1} a_{jj}^{l} E_{j-1,i-l,i-l+1} = 0. \tag{20}$$

Now let a be a parameter, and in (18) set $i=\mu-k$,

$$E_{j,\mu-k,\mu-k+1} - E_{j-1,\mu-k,\mu-k+1} - a_{jj}E_{j,\mu-k-1,\mu-k} = 0,$$

and multiply this by a^k and sum with respect to k:

$$\sum_{k=1}^{\mu} a^k(E_{j,\mu-k,\mu-k+1} - E_{j-1,\mu-k,\mu-k+1}) - a_{jj}\sum_{k=0}^{\mu} a^kE_{j,\mu-k-1,\mu-k} = 0.$$

To this add the negative of (20) with μ substituted for i, and then multiplying the identity

$$E_{j,-1,0} - E_{j-1,-1,0} = 0$$

by $a^{\mu+1}$ add it in also, thus obtaining

$$\sum_{k=1}^{\mu+1} a^k(E_{j,\mu-k,\mu-k+1} - E_{j-1,\mu-k,\mu-k+1}) - a_{jj}\sum_{k=0}^{\mu} a^kE_{j,\mu-k-1,\mu-k}$$
$$+ \sum_{k=1}^{\mu+1} \overset{k}{a}_{jj}E_{j-1,\mu-k,\mu-k+1} = 0.$$

In the second summation set $k'=k+1$; then since k' is a variable of summation the prime may be dropped and the terms rearranged in the form

$$\sum_{k=1}^{\mu+1}\left[(a^k - a_{jj}a^{k-1})E_{j,\mu-k,\mu-k+1} - (a^k - \overset{k}{a}_{jj})E_{j-1,\mu-k,\mu-k+1}\right] = 0.$$

Separate this into two sums, and in the first replace $k-1$ by k:

$$\sum_{k=0}^{\mu}(a^{k+1} - a_{jj}a^k)E_{j,\mu-k-1,\mu-k} - \sum_{k=1}^{\mu+1}(a^k - \overset{k}{a}_{jj})E_{j-1,\mu-k,\mu-k+1} = 0\ ;$$

then divide this by $a-a_{jj}$ and set $\mu=t-j$, obtaining finally

$$\sum_{k=0}^{t-j} a^kE_{j,t-j-k-1,t-j-k} = \sum_{k=1}^{t-j+1}\left(\frac{a^k - \overset{k}{a}_{jj}}{a - a_{jj}}\right)E_{j-1,t-j-k,t-j-k+1}, \tag{21}$$

which is the identity that will be needed later.

In particular note from (17a) that

$$E_{1,i,i+1} = E_{1,i-1,i}a_{11}\ ;$$

but $E_{1,0,1}=a_{11}$, whence it follows at once that

$$E_{1,i-1,i} = \overset{i}{a}_{11}. \tag{22}$$

In order to study the manner in which the a's enter the E's it is convenient to say that $a_{s+q,s}$ is earlier than $a_{t+r,t}$ if $q<r$, and also that $a_{s+q,s}$ is earlier than $a_{r+q,r}$ if $s<r$. Thus the a's are arranged in order.

From the work above it is clear that $E_{j,\sigma,\sigma+1}$ contains of the a's only $a_{11}, a_{22}, \cdots, a_{jj}$. Consider next

$$E_{j,i,i} = \sum_{s=1}^{j} \left\{ \sum_{r=s}^{s+1} E_{r,s+i-r-1,i-1} a_{rs} + E'_{s,i-1,i} \right\}$$

$$(23) \qquad = \sum_{s=1}^{j} \left\{ E_{s,i-1,i-1} a_{ss} + E_{s+1,i-2,i-1} a_{s+1,s} + E'_{s,i-1,i} \right\},$$

in which the derivative term is of the form that contains only the a_{jj}. In particular

$$E_{j,1,1} = \sum_{s=1}^{j} \left\{ E'_{s,0,1} + a_{s+1,s} \right\},$$

and so contains only the a_{ii} and $a_{i+1,i}$ for values of i in the range $1 \leqq i \leqq j$. Hence by (23) $E_{j,i,i}$ contains the a_{ii} and their derivatives and $a_{i+1,i}$ for the same range of values of i.

A similar argument can be applied to show that $E_{j,i,k}$ contains $a_{j+i+1-k,j}$ and earlier a's and their derivatives, and the order of the derivative of $a_{r+s,r}$ occurring is not greater than $i+1-k-s$.

It is now necessary to determine the coefficient of this latest a, namely $a_{j+i+1-k,j}$, in the expression for $E_{j,i,k}$. By reference to (17) it will be seen that this a occurs only in the two terms

$$a_{j+i-k+1,j} E_{j+i-k+1,k-2,k-1} + a_{jj} E_{j,i-1,k-1}.$$

The first E is of the form that contains only the a_{ii}. By use of the same formula the terms in $E_{j,i-1,k-1}$ that contain $a_{j+i-k+1,j}$ may then be found. Each repetition of this procedure lowers the second and last subscripts of the last E by unity, so that finally the coefficient of the latest a, namely $a_{j+i-k+1,j}$, in $E_{j,i,k}$ is

$$(24) \qquad \sum_{l=0}^{k-1} \overset{l}{a}_{jj} E_{j+i-k+1,k-l-2,k-l-1}.$$

It has already been noted that the differential equation (9) by reason of (13) has coefficients of the form in ρ prescribed for (1). In order to prove the theorem of this paper it is then necessary to show how the a's and b's of the first-order system may be determined in terms of the P's of the nth order equation. This is done by means of the relation (10), which because of (13) may be written

$$(25) \qquad P_{j-1} + \sum_{t=j}^{n} P_t C_{tj} = 0 \qquad (j = 1, 2, \cdots, n).$$

By substitution from (2) and (16) in (25),

$$\sum_{\mu=0}^{n-j+1} P_{n-j+1,\mu}\rho^{\mu} = \sum_{t=j}^{n}\left(\sum_{\mu=0}^{n-t} P_{n-t,\mu}\rho^{\mu}\right)\left(\sum_{k=1}^{t-j+1} E_{j,t-j,k}\rho^{k}\right) + b_{nj} = 0,$$

from which it follows that

$$P_{n-j+1,0} = -\,b_{nj} \qquad (j = 1, 2, \cdots, n),$$

so that the b's are determined.

The coefficient of ρ^{ν} above gives then

$$P_{n-j+1,\nu} + \sum_{t=j}^{n}\sum_{\mu=0}^{n-t} P_{n-t,\mu}E_{j,t-j,\nu-\mu} = 0 \tag{26}$$

$$(j = 1, 2, \cdots, n;\ \nu = 1, \cdots, n-j+1).$$

Consider first the equation in which ν has its largest value, namely

$$P_{n-j+1,n-j+1} + \sum_{t=j}^{n} P_{n-t,n-t}E_{j,t-j,t-j+1} = 0. \tag{27}$$

In particular for $j=1$ this gives

$$P_{nn} + \sum_{t=1}^{n} P_{n-t,n-t}E_{1,t-1,t} = 0.$$

But by (22) this becomes

$$P_{nn} + \sum_{t=1}^{n} P_{n-t,n-t}a_{11}^{t} = 0,$$

which shows that a_{11} is a root of the characteristic equation. Denote by $f_j(a_{jj})$ the expression (27), and by $f_j(a)$ the expression formed from this by substituting a for a_{jj}. By (27) and (19),

$$f_j(a_{jj}) = P_{n-j+1,n-j+1} + \sum_{t=j}^{n} P_{n-t,n-t}\sum_{k=0}^{t-j+1} a_{jj}^{k}E_{j-1,t-j-k,t-j-k+1},$$

in which the E's now occurring do not contain a_{jj}. Hence

$$f_j(a) = P_{n-j+1,n-j+1} + \sum_{t=j}^{n} P_{n-t,n-t}\sum_{k=0}^{t-j+1} a^{k}E_{j-1,t-j-k,t-j-k+1}\cdot$$

and from these two expressions is obtained

$$
(28)\qquad
\begin{aligned}
\frac{f_j(a) - f_j(a_{jj})}{a - a_{jj}} &= \sum_{t=j}^{n} P_{n-t,n-t} \sum_{k=0}^{t-j+1} \frac{a^k - a_{jj}^k}{a - a_{jj}} E_{j-1,t-j-k,t-j-k+1} \\
&= P_{n-j,n-j} + \sum_{t=j+1}^{n} P_{n-t,n-t} \sum_{k=1}^{t-j+1} \left(\frac{a^k - a_{jj}^k}{a - a_{jj}} \right) E_{j-1,t-j-k,t-j-k+1}.
\end{aligned}
$$

On the other hand

$$
f_{j+1}(a_{j+1,j+1}) = P_{n-j,n-j} + \sum_{t=j+1}^{n} P_{n-t,n-t} \sum_{k=0}^{t-j} a_{j+1,j+1}^k E_{j,t-j-k-1,t-j-k},
$$

and so

$$
(29)\qquad f_{j+1}(a) = P_{n-j,n-j} + \sum_{t=j+1}^{n} P_{n-t,n-t} \sum_{k=0}^{t-j} a^k E_{j,t-j-k-1,t-j-k}.
$$

But by (21) the right hand sides of the equations (28) and (29) are equal, so that

$$
\frac{f_j(a) - f_j(a_{jj})}{a - a_{jj}} = f_{j+1}(a),
$$

and it follows at once that the a_{jj} are the roots of the characteristic equation $f_1(a) = 0$.

The earliest of the a's are thus determined. The rest of the proof of the theorem then consists in showing how any a may be determined in terms of P's and earlier a's. By reference to (26) it is seen that the latest a is contained in the E's that have the greatest difference between the last two subscripts, that is, in those for which $t - j - \nu + \mu$ is a maximum, which it is for $\mu = n - t$. It is then contained in the terms

$$
\sum_{t=n+1-\nu}^{n} P_{n-t,n-t} E_{j,t-j,\nu+t-n},
$$

and by using (24) it is seen that the coefficient of this latest a, namely $a_{n+1-\nu,j}$, is

$$
(30)\qquad \sum_{t=n+1-\nu}^{n} P_{n-t,n-t} \sum_{l=0}^{k-1} a_{jj}^l E_{n+1-\nu,k-l-2,k-l-1}
$$

where $k = \nu + t - n$.

Thus the equation (26) is linear in the latest a occurring in it, with the coefficient of this a given by (30). If then it can be shown that this coefficient is not zero the proof of the theorem will be complete. Now set $n + 1 - \nu = i$, and (30) becomes

$$\sum_{t=i}^{n} P_{n-t,n-t} \sum_{l=0}^{t-i} a_{jj}^{l} E_{i,t-i-l-1,t-i-l},$$

which may be written

$$P_{n-i,n-i} + \sum_{t=i+1}^{n} P_{n-t,n-t} \sum_{l=0}^{t-i} a_{jj}^{k} E_{i,t-i-l-1,t-i-l},$$

and by (29) this is $f_{i+1}(a_{jj})$, and so the coefficient (30) is

$$f_{n-\nu+2}(a_{jj}) = \prod_{i=n-\nu+2}^{n} (a_{ii} - a_{jj}),$$

which cannot be zero for any value of x in the interval (a, b). Thus the proof is complete.

To find the number of derivatives that any $P_{i,j}$ must possess, apply to formula (26) the known facts concerning the manner in which the E's involve the a's. It is thus found that the determination of $a_{j+k,j}$ involves derivatives of $a_{r+s,r}$ of order not greater than $k-s$. Again by referring to (26) it will be seen that $P_{t+s,t}$ is not involved in the determination of any a earlier than $a_{r+s,r}$, so that in the determination of $a_{j+k,j}$ its derivatives will not occur to an order greater than $k-s$. Since the maximum number of derivatives of any $P_{t+s,t}$ will occur in the determination of the latest of the a's, namely a_{n1}, the function $P_{t+s,t}$ will not need to be differentiated more than $n-1-s$ times.

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ON THE "THIRD AXIOM OF METRIC SPACE"*

BY

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1. In his thesis Fréchet† defines a *metric space*‡ or class (D) as a class of elements, the relations among which are established by means of a function of pairs of elements of this class. For any two elements x and y of class (D) this function $\rho(x, y)$ must satisfy three requirements, which we shall call *the axioms of metric space.*

Axiom I. (Coincidence Axiom.) $\rho(x, y)=0$, *when and only when* $x=y$.

Axiom II. (Axiom of Symmetry.) $\rho(x, y)=\rho(y, x)$.

Axiom III. (Triangle Axiom.) $\rho(x, y)=\rho(y, z)+\rho(z, x)$.

It is evident that the metric spaces are cases of the topological spaces of Hausdorff.§ In addition, Fréchet has considered spaces or classes of elements which he calls classes (E). These are classes of elements, the relations among which are established by means of a function $\delta(x, y)$ satisfying the coincidence axiom and the axiom of symmetry. A class (E) which is also a topological space of Hausdorff will be called a *symmetric space.*

It is the purpose of the present paper to present a generalization of the seventh theorem of a joint paper by A. D. Pitcher and E. W. Chittenden‖ concerning the investigation of Axiom III. These authors considered spaces defined by functions $\delta(x, y)$ satisfying Axioms I and II together with one or more of the following three conditions:

(1) (Ch)¶ $\lim \delta(x_n, x) = 0,\quad \lim \delta(x_n, y_n) = 0,\quad$ imply $\lim \delta(y_n, x) = 0$;

(2) $\lim \delta(x_n, x) = 0,\quad \lim \delta(y_n, x) = 0,\quad$ imply $\lim \delta(x_n, y_n) = 0$;

(3) $\lim \delta(x_n, y_n) = 0,\quad \lim \delta(y_n, z_n) = 0,\quad$ imply $\lim \delta(x_n, z_n) = 0$;

where $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are sequences of elements of the given class (E). The theorem cited may be stated in the following form: A *compact* coherent class (E) is a compact metric space.**

* Presented to the Society, September 9, 1926; received by the editors in September, 1926.

† *Sur quelques points du calcul fonctionnel*, Rendiconti del Circolo Matematico di Palermo, vol. 22 (1906).

‡ The term is due to Hausdorff; *Grundzüge der Mengenlehre*, Leipzig, Veit, 1914, p. 211.

§ Loc. cit., p. 213.

‖ These Transactions, vol. 19 (1918).

¶ A space in which this condition is satisfied is *coherent* in the terminology of Pitcher and Chittenden.

** That is, it is possible to define in terms of the given function $\delta(x,y)$ an infinitesimally equivalent function $\rho(x, y)$ which satisfies the three metric axioms.

The theorem to be proved is as follows.

THEOREM. *A symmetric space in which the condition* (Ch) *is satisfied is a metric space.*

I shall give two demonstrations of this theorem: one, based entirely on the method developed by Pitcher and Chittenden; another, based on the methods and results of the Russian school.

2. LEMMA. *If the condition* (Ch) *is satisfied in a class* (E) *the following condition is also satisfied:**

$$\lim \delta_1(x_n, y_n) = 0, \quad \lim \delta_1(y_n, z_n) = 0, \quad \textit{imply} \lim \delta_1(x_n, z_n) = 0.$$

Let x and y be two arbitrary elements of the given class(E). Let $\delta(x, y) = \eta$. Consider the following: (1) the points of type z' satisfying the conditions

$$\delta(x, z') \leqq \eta, \quad \delta(y, z') > 2\eta\ ;$$

and set $d_1 = \lim\sup \delta(y, z')$; (2) the points of type z'' satisfying the conditions

$$\delta(y, z'') \leqq \eta, \quad \delta(x, z'') > 2\eta\ ;$$

and set $d_2 = \lim\sup \delta(x, z'')$.† If we write $d_0 = (d_1 + d_2)/2$, it follows immediately that $d_0 \geqq \eta$.

Denote d_0 by $\delta_1(x, y)$. It is evident that $\delta_1(x, y) \geqq \delta(x, y)$. Then, to show the equivalence of the functions $\delta_1(x, y)$, $\delta(x, y)$, it is sufficient to prove that

$$\lim \delta(x_n, x) = 0 \quad \text{implies} \quad \lim \delta_1(x_n, x) = 0.$$

Suppose this is not true. Then there is an element x and a sequence $\{x_n\}$ such that

$$\lim \delta(x_n, x) = 0, \quad \delta_1(x_n, x) \geqq \eta > 0 \quad (n = 1, 2, 3, \cdots).$$

Therefore by the definition of $\delta_1(x, y)$ there exist points z_n which are either of type z' for infinitely many integers n or of type z''.

If infinitely many of the points z are of type z', a sequence $\{z_{n_i}\}$ exists such that

$$\lim \delta(x_{n_i}, z'_{n_i}) = 0, \quad \lim \delta(x_{n_i}, x) = 0,$$
$$\delta(z'_{n_i}, x) \geqq \eta > 0 \qquad (i = 1, 2, 3, \cdots).$$

But this contradicts the condition of the lemma.

---

* Relative to a different function $\delta_1(x, y)$, equivalent to $\delta(x,y)$.

† In case there are no points z', z'' of these types, let $\delta_1(x, y) = \delta(x, y)$.

If infinitely many of the points z_n are of the type z'' there exists a sequence $\{z''_{n_i}\}$ such that

$$\lim \delta(x_{n_i}, x) = 0,\ \lim \delta(z''_{n_i}, x) = 0,\ \delta(x_{n_i}, z''_{n_i}) \geqq \eta > 0 \qquad (i = 1, 2, 3, \cdots).$$

But this is also impossible by condition (Ch) and a theorem of Pitcher and Chittenden.*

The equivalence of the functions $\delta_1(x, y)$ and $\delta(x, y)$ is established.

It will now be shown that the function $\delta_1(x, y)$ satisfies the condition of the lemma. Suppose that it does not, then two cases may occur.

(i) There are sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$, which do not satisfy the requirement of the theorem either by the old definition of the distance function or by the new one. Then we have

$$\lim \delta(x_n, y_n) = 0, \quad \lim \delta(y_n, z_n) = 0, \quad \delta(x_n, z_n) > \eta \qquad (n = 1, 2, 3, \cdots);$$

$$\lim \delta_1(x_n, y_n) = 0, \quad \lim \delta_1(y_n, z_n) = 0, \quad \delta_1(x_n, z_n) > \eta \qquad (n = 1, 2, 3, \cdots).$$

Let us take N large enough to have

$$\delta_1(x_n, y_n) \leqq n/2, \qquad \delta_1(y_n, z_n) \leqq n/2 \qquad (n \geqq N).$$

The same inequalities are satisfied by $\delta(\leqq \delta_1)$, that is,

$$\delta(x_n, y_n) \leqq n/2, \quad \delta(y_n, z_n) \leqq n/2 \qquad (n \geqq N).$$

There are two sub-cases:

(1) $\delta(x_n, y_n) \leqq \delta(y_n, z_n)$; (2) $\delta(x_n, y_n) > \delta(y_n, z_n)$.

In the first case we denote $\delta(y_n, z_n)$ by ϵ' and obtain

$$\delta(y_n, z_n) = \epsilon', \ \delta(x_n, y_n) \leqq \epsilon', \ \delta(x_n, z_n) > \eta \geqq 2\epsilon'.$$

It then follows from the definition of the function $\delta_1(x, y)$ that $\delta_1(x_n, y_n) > \eta$, a contradiction.

If we denote $\delta(x_n, y_n)$ in the second case by ϵ'' a similar contradiction is obtained.

(ii) There exist sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, which satisfy the conditions of the theorem by the old definition of the distance function, but not by the new one. Then we should have

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* Loc. cit., Theorem 1. This theorem is equivalent to the following statement: In every coherent class (E) it is possible to define the function $\delta(x, y)$ so that condition (2) is satisfied.

$$\lim \delta(x_n, y_n) = 0, \quad \lim \delta(y_n, z_n) = 0, \quad \lim \delta(x_n, z_n) = 0 ;$$

$$\lim \delta_1(x_n, y_n) = 0, \quad \lim \delta_1(y_n, z_n) = 0, \quad \delta_1(x_n, z_n) > \eta > 0 \qquad (n = 1, 2, 3, \cdots).$$

It follows that a sequence $\{w_{n_i}\}$ exists such that

$$\lim \delta(x_{n_i}, w_{n_i}) = 0, \quad \delta(w_{n_i}, z_{n_i}) > 0 > \eta \quad (i = 1, 2, 3, \cdots),$$

or vice versa. Because of the complete symmetry, we need consider only the first case. About the sequence $\{w_{n_i}\}$ we can make the two following suppositions:

$$\text{(1)} \qquad \lim \delta(w_{n_i}, y_{n_i}) = 0 ; \quad \text{(2)} \ \delta(w_{n_i}, y_{n_i}) > \eta > 0 \ (i = 1, 2, 3, \cdots).$$

Let us consider them separately. In the first case it follows readily from the definition of $\delta_1(x, y)$ that

$$\limsup \delta_1(y_{n_i}, z_{n_i}) \geqq \eta > 0,$$

which contradicts the hypothesis. In the second case we have similarly

$$\limsup \delta_1(x_{n_i}, y_{n_i}) \geqq \eta > 0.$$

This completes the proof of the lemma.

Pitcher and Chittenden have proved that the distance function of the lemma is equivalent to a uniformly regular écart.* It therefore follows from the lemma and the fundamental result of Chittenden† on the equivalence of "écart régulier" and distance that the theorem proposed is completely proved.‡

3. In this section the theorem of § 1 is demonstrated by means of new methods and with a new formulation of special interest.

Definition. *A topological space R satisfies the local axiom of the triangle, if for every element x and positive number* ϵ *a number* η_x *may be found such that*

$$\text{(F)} \qquad \delta(x, y) \leqq \eta_x, \quad \delta(x, y) \leqq \eta_x \ \textit{imply} \ \delta(x, z) \leqq \epsilon.$$

To facilitate the following proof we write the condition (F) in the following form.

---

* Theorem 4, loc. cit.

† These Transactions, vol. 18 (1917).

‡ In the statement of this theorem we have supposed that the class (E) is topological in the sense of Hausdorff, but we have not used this condition. It is not difficult to prove the following theorem: *A class* (E) *satisfying the condition* (Ch) *is a topological space of Hausdorff.*

For every element x and positive number ϵ there exists an η_x and a region G_x such that if $\delta(x, y) < \eta_x$ and $\delta(y, z) < \eta_x$, then*

$$z \subset G_x \subset S(x,\epsilon).\dagger \tag{N}$$

THEOREM. *Every symmetric space R in which the local axiom of the triangle is satisfied is a metric space.*

From condition (N) there exists for every point of the space R a sequence of spheres

$$S_1^x, S_2^x, S_3^x, \cdots, S_k^x = S(x,\epsilon_k^x), \cdots \qquad (\lim \epsilon_k^x = 0),$$

satisfying the following condition:

(B) *Every sphere S_i^x contains a region G_i^x such that for every point y of S_{i+1}^x, $S(y, \epsilon_{i+1}^x) \subset G_i^x$.*

The families of regions

$$\Pi_1 = (G_1^x), \Pi_2 = (G_2^x), \cdots, \Pi_k = (G_k^x), \cdots$$

form a sequence of coverings of the space R whose properties we shall investigate.‡

Let z be an element of $G_k^x \cdot G_k^y$. Then for the corresponding spheres S_k^x and S_k^y, we have $z \subset S_k^x \cdot S_k^y$. Let the radius of S_k^x be γ_1, and the radius of S_k^y be γ_2, and let the notation x, y be chosen so that $\gamma_2 \leqq \gamma_1$.

By the axiom of symmetry we have

$$\delta(z,x) < \gamma_1, \qquad \delta(z,y) < \gamma_2 \leqq \gamma_1.$$

If we describe about the point z a sphere of radius γ_1 it will contain the points x and y. From condition (B) we have

$$S_{k-1}^x \supset S_k^x + S(z,\gamma_1).$$

Let the radius of S_{k-1}^x be γ_3 $(\gamma_2 \leqq \gamma_1 < \gamma_3)$. Describe about the point y a sphere of radius γ_3. This sphere will by construction include the sphere S_k^y. A second application of condition (B) shows that

$$S_{k-2}^x \supset S_{k-1}^x + S(y,\gamma_3).$$

---

* By a *region* we understand the complement of a closed set.

† Consider the sphere $S(x, \epsilon)$ of center x and radius ϵ. As the space R is topological in the sense of Hausdorff, there exists a region G_x which contains x and is a subset of $S(x, \epsilon)$. Likewise, G_x contains a sphere $S(x, \epsilon')$. Let us choose, in accordance with condition (F), a number η_x such that if $\delta(x, y) \leqq \eta_x$, $\delta(y, z) \leqq \eta_x$, then $\delta(x, z) < \epsilon'$. The number η_x evidently satisfies condition (N).

‡ P. Alexandroff and P. Urysohn (Comptes Rendus, vol. 177, p. 1274) have defined a *covering* as a collection of regions such that every point of the space belongs to at least one of them.

Furthermore, there exists a region G_{k-2}^{x} such that

(K) $$S_{k-2}^{x} \supset G_{k-2}^{x} \supset S_k^{x} + S_k^{y} \supset G_k^{x} + G_k^{y}.$$

We now consider the chain of coverings

$$\Pi_1', \Pi_2', \cdots, \Pi_k', \cdots$$

obtained by setting

$$\Pi_k' = \Pi_{2k+1}.$$

It will be shown that this chain is regular and complete.* That the chain is regular follows from its construction. Let us prove it complete.

Let $G_1, G_2, \cdots, G_k, \cdots$ be a sequence of regions such that G_k contains a point x for every value of k, and G_k belongs to the covering Π_k'. Since each of the sets G_k is a region, every G_k contains a neighborhood of the point x. Suppose that U is a neighborhood of x which contains no set G_k. Consider the sequence of spheres $S_k^{y_k}$ of radius ϵ_k ($\lim \epsilon_k = 0$), such that

$$G_k^{y_k} \subset S_k^{y_k}.$$

Then $\lim \delta(x, y) = 0$, obviously, and therefore y_k must, for sufficiently large values of k, belong to U. Let z_k be a point of G_k, which does not belong to U. Since U is a neighborhood of the point x it contains a sphere $S(x, \epsilon)$. We have $\delta(z_k, x) \geqq \epsilon$. But $\lim \delta(x, y_k) = 0$, $\lim \delta(y_k, z_k) = 0$. This contradicts condition (F).

Since P. Alexandroff and P. Urysohn have shown† that every topological space of Hausdorff which admits a complete and regular chain of coverings is a metric space the proof of the theorem is complete.

To complete the proof of the theorem of § 1 it is required to show that the condition (Ch) is equivalent to the local axiom of the triangle.

If condition (Ch) is fulfilled, condition (F) is also satisfied. Otherwise there exist a point x and a positive number ϵ, such that for every small positive number η there will exist points y and z satisfying the inequalities

$$\delta(x, y) < \eta, \quad \delta(y, z) < \eta, \quad \delta(x, y) \geqq \epsilon.$$

* P. Alexandroff and P. Urysohn (loc. cit.) have called a chain of coverings *complete* if for any point x of the space R, and any regions $V_1, V_2, \cdots, V_n, \cdots$ containing x and belonging to $\Pi_1, \Pi_2, \cdots, \Pi_n, \cdots$ respectively the sequence $\{V_n\}$ defines the point x in R, that is, it is a complete system of neighborhoods of the point x; and a chain of coverings *regular* if the following condition is fulfilled: for every integer n and for arbitrary regions V_n and W_n of a covering Π_n there exists in Π_{n-1} a region V_{n-1} containing both V_n and W_n.

† Loc. cit.

Choose $\eta_1, \eta_2, \eta_3, \cdots, \eta_n, \cdots$ ($\lim \eta_n = 0$), $y_1, y_2, \cdots, y_n, \cdots$ and $z_1, z_2, \cdots, z_n, \cdots$ correspondingly, so that we will have

$$\limsup \delta(x, y_n) = 0, \quad \limsup \delta(y_n, z_n) = 0, \quad \limsup \delta(x, z_n) \geqq \epsilon.$$

Then the condition (Ch) is not satisfied as we assumed.

Suppose that condition (F) is satisfied and that condition (Ch) is not. Then there are a point x, a positive number ϵ, and a pair of sequences $\{y_n\}$, $\{z_n\}$ of points such that

$$\limsup \delta(x, y_n) = 0, \quad \limsup \delta(y_n, z_n) = 0, \quad \limsup \delta(x, z_n) \geqq \epsilon.$$

For this value of ϵ one cannot choose a number η_x to satisfy condition (F), contrary to hypothesis.

This completes the proof.

But the investigation of the third axiom is not completed, for those topological conditions which imply the local axiom of the triangle are left undetermined. I intend to investigate this question more closely in my next paper.

Moscow, Russia

IMPLICIT FUNCTIONS AND DIFFERENTIAL EQUATIONS IN GENERAL ANALYSIS*

BY

LAWRENCE M. GRAVES†

The chief purpose of this paper is to discuss some special cases of the implicit function theorems obtained by Hildebrandt and Graves in the paper entitled *Implicit functions and their differentials in general analysis*.‡ In particular I wish to discuss a generalization of the notion of differential equation, combining ideas due to Hahn and to Carathéodory.§ The equations are of the form

$$f\left(r,\int_{r_0}^{r} g(r,r',y(r'),x)dr',y(r),x\right)=0. \tag{1}$$

Here x may represent both initial values and parameters, and the functions f, g, and y are supposed to be bounded and measurable in r and r'. Imbedding and existence theorems for equations of this form are obtained in Parts VI and VII. As indicated by the paper of Hahn just referred to, such theorems find important applications in the calculus of variations. Special theorems relating to linear equations are found in Part VIII.

Many other special cases of our general theory have appeared in the literature from other writers.‖ In some of these cases it naturally occurs that the hypotheses of the writers must be strengthened in order to make their theories fit under our general theory. On the other hand, we should expect that only a limited part of any special theory will flow directly from

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† National Research Fellow in Mathematics.

‡ These Transactions, vol. 29 (1927), pp. 127–153. In the sequel this paper will be referred to as Paper I. Frequent reference will also be made to Paper II, Graves, *Riemann integration and Taylor's theorem in general analysis*, these Transactions, vol. 29 (1927), pp. 163–177; and Paper III, Graves, *Some theorems concerning measurable functions*, Bulletin of the American Mathematical Society, vol. 32 (1926), p. 529.

§ See Hahn, Monatshefte für Mathematik und Physik, vol. 14 (1903), pp. 326–332.

Carathéodory, *Vorlesungen über reelle Funktionen*, p. 666.

‖ Cf. Hart, these Transactions, vol. 18 (1917), p. 125; vol. 23 (1922), p. 30; Annals of Mathematics, (2), vol. 24 (1922), p. 23.

Barnett, American Journal of Mathematics, vol. 44 (1922), p. 172.

Bliss, these Transactions, vol. 21 (1920), p. 79.

Cotton, Bulletin de la Société Mathématique de France, vol. 38 (1910), p. 144.

our general theory. Such is obviously the case, for example, with regard to the theory of the Fredholm or the Volterra equation of the second kind. However, we have succeeded in obtaining a much more complete theory of the differentiability of the solutions than is exhibited in the previous treatments of many of the special cases.

It is found to be economical of time and effort, even in specializing our general theory, to state theorems that are still very general. Thus a large part of the paper is still concerned with general ranges and abstract spaces.

Part I lists some examples (mostly abstract) of linear metric spaces. Each space consists of a class of functions on a range $\mathfrak{P}$ to a linear metric space. Part II contains some miscellaneous theorems on differentials and compact sets. Part III discusses the nature of the solution of the equation $G(x, y)=y_*$ on subspaces $\mathfrak{X}'$ and $\mathfrak{Y}'$ of the main spaces $\mathfrak{X}$ and $\mathfrak{Y}$. The theorems of Part III are applied in Part VI to discuss the continuity in r of the solutions of equation (1). Part IV discusses a special mode of dependence of the function G on x and y, and obtains as a special case an extended implicit function theorem generalizing those given by Bolza, Bliss and Mason, Hadamard, and Hobson.* Part V discusses a method of extending the domain of definition of a function so as to preserve a Lipschitz condition. Parts VI and VII discuss imbedding and existence theorems for "differential" equations in general analysis analogous to equation (1). Equation (1) is taken up as a special case. Part VIII discusses special properties of the solutions of linear equations of the form

$$K_1(y(r),r) = \int_{r_0}^{r} K_2(y(r),r)dr + c.$$

My acknowledgments are due to Professor Hildebrandt for some of the theorems, which are noted in the text.

I. Examples of linear metric spaces

It is worth noting that all the spaces considered here, as well as the examples given by other writers (cf. Fréchet, Banach, Hahn) have as elements, functions on a range $\mathfrak{P}$ to a given linear metric space. Professor M. H. Ingraham informs me that every linear metric space which can be well-ordered, is isomorphic with a class of numerically valued functions.

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* See Bolza, *Vorlesungen über Variationsrechnung*, p. 160–168.
Bliss and Mason, *Fields of extremals in space*, these Transactions, vol. 11 (1910), p. 326.
Hadamard, *Leçons sur le Calcul des Variations*, vol. I, pp. 497–502.
Hobson, Proceedings of the London Mathematical Society, vol. 14 (1915), p. 147.

It is also worth remarking that the property of being *complete* remains isolated from the other properties. That is to say, in spaces of type A or A_0, completeness is added by a separate postulate, and in spaces of type B, C_0, or C_n, completeness is present if it is so in the range of functional values. All the types of spaces mentioned are special cases of type A.

We follow the practice of Paper I in denoting by $\mathfrak{P}$ and $\mathfrak{P}'$ general ranges or classes (not empty), and by σ and σ', numerically valued functions on $\mathfrak{P}$ and $\mathfrak{P}'$ respectively, which will enter as scale functions.

1. Postulationally limited classes. We have the following types.

Type A. A system $(\mathfrak{P}, \mathfrak{B}, \mathfrak{A}, \mathfrak{Y}, \|\,\|)$, satisfying conditions 1.1–1.9:

1.1. $\mathfrak{B}$ is a linear metric space.

1.2. $\mathfrak{A}$ is the number system for $\mathfrak{B}$.

1.3. $\mathfrak{Y}$ is a class of functions on $\mathfrak{P}$ to $\mathfrak{B}$, containing at least two functions.

1.4. The sum function, y_1+y_2, is contained in $\mathfrak{Y}$, for every y_1 and y_2.

1.5. The product function, ya, is contained in $\mathfrak{Y}$, for every element y of $\mathfrak{Y}$ and every number a.

1.6. $\|\,\|$ is a function on $\mathfrak{Y}$ to the real non-negative part of $\mathfrak{A}$.

1.7. $\|y\|=0$ implies $y(p)=v^*$ for every p, that is, $y = y^*$.

1.8. $\|y_1+y_2\| \leqq \|y_1\|+\|y_2\|$ for every y_1 and y_2.

1.9. $\|ya\|=\|y\|\ |a|$ for every y and a.

The sum of two functions and the product of a function by a number are defined in the customary way. Then postulates 1.4 and 1.5 express the linear closure of the class $\mathfrak{Y}$. We use the same sign $\|\,\|$ for the norm or modulus in the spaces $\mathfrak{B}$ and $\mathfrak{Y}$. This can bring no confusion, since $y(p)$ denotes an element of the space $\mathfrak{B}$, viz., the functional value of the function y at p, so that by $\|y(p)\|$ we understand the norm in the space $\mathfrak{B}$, and by $\|y\|$ the norm in the space $\mathfrak{Y}$.

Type A_0. Let $\mathfrak{Y}$ be a class of functions y on $\mathfrak{P}$ to $\mathfrak{B}$, satisfying postulates 1.1 to 1.5 for type A, and let σ be a scale function on $\mathfrak{P}$ to $\mathfrak{A}$. Suppose each function y of $\mathfrak{Y}$ is bounded $(\mathfrak{P};\sigma)$, i.e., for each y there exists a constant M_y such that $\|y(p)\| \leqq M_y|\sigma(p)|$ for every p. Then if we set $\|y\|=$minimum effective M_y, $\mathfrak{Y}$ is a space of type A.

2. Classes of all functions satisfying certain conditions. We have the following types.

Type B. Corresponding to a given $\mathfrak{P}$, $\mathfrak{B}$, and σ, let $\mathfrak{Y}$ be the class of all functions y on $\mathfrak{P}$ to $\mathfrak{B}$ which are bounded $(\mathfrak{P};\sigma)$. Then $\mathfrak{Y}$ is a space of type A_0, if $\|y\|$ is defined as above. If $\mathfrak{B}$ is complete, so is $\mathfrak{Y}$.

Spaces of types C_0 and C_n are subspaces of spaces of type B. Note that to secure completeness in a space of type C_n, the definition of the norm function $\| \ \|$ must be altered.

Type C_0. Let $\mathfrak{X}_0$ be a region of a metric space $\mathfrak{X}$, and let $\mathfrak{P}=\mathfrak{X}_0\mathfrak{P}'$. Let the scale function σ be independent of x, and let $\mathfrak{Y}$ be the corresponding space of type B. Let $\mathfrak{Y}^{(0)}$ be the subspace of $\mathfrak{Y}$ composed of all those functions y which are continuous on $\mathfrak{X}_0$ uniformly $([\mathfrak{X}_0]\mathfrak{P}'; \sigma)$. (The square brackets around $\mathfrak{X}_0$ indicate that it may be omitted from the range of uniformity.) Then $\mathfrak{Y}^{(0)}$ is a linear metric space of type A_0, and if $\mathfrak{B}$ is complete, so is $\mathfrak{Y}^{(0)}$.

Type C_n. Let the space $\mathfrak{X}$ be *linear* metric, and let $\mathfrak{Y}^{(n)}$ be the subspace of $\mathfrak{Y}$ composed of all those functions y which are of class $\mathfrak{C}^{(n)}$ on $\mathfrak{X}_0$ uniformly $([\mathfrak{X}_0]\mathfrak{P}'; \sigma)$. Then $\mathfrak{Y}^{(n)}$ is a linear metric space of type A_0, if the function $\| \ \|$ is unaltered.

The conditions for completeness, however, are now more complicated. Consider first the case when $\mathfrak{X}_0$ is omitted from the range of uniformity for the differentiability properties. We assume that the space $\mathfrak{B}$ is complete, and restrict the space $\mathfrak{Y}^{(n)}$ further by supposing that for each function y, all the differentials up to and including the nth differential are modular uniformly $(\mathfrak{X}_0\mathfrak{P}'; \sigma)$. Furthermore we alter the norm or distance function as follows: let $N(y)$ denote the norm $\|y\|$ of the function y regarded as a point of the space $\mathfrak{Y}$ of type B, and let $M_1(y), M_2(y), \cdots, M_n(y)$ denote the moduli of the first n differentials of y, respectively. Then $\|y\|$ is now defined to be the greatest of the numbers $N(y), M_1(y), \cdots, M_n(y)$. It is readily verified that $\mathfrak{Y}^{(n)}$ is still a linear metric space. To show that it is complete, we proceed as follows.

Consider first the case $n=1$. Let $\{y_m\}$ be a sequence of functions of the space $\mathfrak{Y}^{(1)}$ satisfying the condition

$$\lim_{\substack{m_1=\infty \\ m_2=\infty}} \|y_{m_1} - y_{m_2}\| = 0.$$

Then by classic arguments we know that there exists a function y of the space $\mathfrak{Y}^{(0)}$ (type C_0) such that

$$\lim_{m=\infty} y_m = y \text{ uniformly } (\mathfrak{X}_0\mathfrak{P}';\sigma).$$

Similarly there exists a function $z(x, p'; dx)$ on $\mathfrak{X}_0\mathfrak{P}'\mathfrak{X}$ to $\mathfrak{B}$ which is continuous on $\mathfrak{X}_0$ uniformly $(\mathfrak{P}'\mathfrak{X}; \sigma\|dx\|)$, modular on $\mathfrak{X}$ uniformly $(\mathfrak{X}_0\mathfrak{P}'; \sigma)$, and such that

$$\lim_{m=\infty} dy_m = z \text{ uniformly } (\mathfrak{X}_0\mathfrak{P}'\mathfrak{X};\sigma\|dx\|)\,.$$

Since z is the limit of functions distributive in dx, z is also distributive in its argument dx. It remains only to verify the third property in the definition of the class $\mathfrak{C}'$. For convenience we now omit to write the argument p. The difference

$$R(x,x_0)\|x - x_0\| = y(x) - y(x_0) - z(x_0; x - x_0)$$

is the sum of the following five terms:

$$\begin{aligned}
R_1 &= y(x) - y_m(x), \\
R_2 &= y_m(x_0) - y(x_0), \\
R_3 &= y_m(x) - y_m(x_0) - y_q(x) + y_q(x_0), \\
R_4 &= y_q(x) - y_q(x_0) - dy_q(x_0; x - x_0), \\
R_5 &= dy_q(x_0; x - x_0) - z(x_0; x - x_0).
\end{aligned}$$

By Taylor's theorem,* we can reduce R_3 to the form

$$R_3 = \int_0^1 [dy_m(x_0 + (x - x_0)r, x - x_0) - dy_q(x_0 + (x - x_0)r, x - x_0)]dr,$$

valid for all points x in a neighborhood $(x_0)_a$ contained in $\mathfrak{X}_0$. From this we obtain the inequality

$$\|R_3\| \leqq \int_0^1 \|dy_m - dy_q\| dr \leqq \epsilon |\sigma| \|x - x_0\|,$$

holding for all values of $m \geqq q$, where q is an integer sufficiently large, and for all points x in $(x_0)_a$. We use also the fact that the convergence of the sequence $\{dy_m\}$ is uniform $(\mathfrak{X}_0\mathfrak{P}'\mathfrak{X}; \sigma\|dx\|)$. For the same q we have

$$\|R_5\| \leqq \epsilon |\sigma| \|x - x_0\|.$$

Since y_q is of class $\mathfrak{C}'$, we have

$$\|R_4\| \leqq \epsilon |\sigma| \|x - x_0\|$$

for all points x in a neighborhood $(x_0)_b$. The integer m is still at our disposal. For each x in $(x_0)_b$ we can choose m so large that

$$\|R_1 + R_2\| \leqq \epsilon |\sigma| \|x - x_0\|.$$

Hence, whenever x is sufficiently near x_0, we have

$$\|R(x,x_0)\| \leqq 4\epsilon |\sigma|,$$

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* See Paper II.

so that the function y is actually of class $\mathfrak{C}'$ on $\mathfrak{X}_0$ uniformly $(\mathfrak{P}'; \sigma)$, and has the function z for its differential.

To prove our statements for all values of n, we use induction. We suppose that $\mathfrak{Y}^{(k)}$ is complete, and that we are concerned with a $\mathfrak{Y}^{(k+1)}$. Then a regular sequence in $\mathfrak{Y}^{(k+1)}$ is regular in $\mathfrak{Y}^{(k)}$, and hence has a limit in $\mathfrak{Y}^{(k)}$. The sequence of $(k+1)$st differentials likewise has a limit, which is shown to be the first differential of the kth differential $d^k y$ of y, with the requisite uniformity, in the same way as the function z above was shown to be dy. The only difference is that the range $\mathfrak{P}'$ is replaced by $\mathfrak{P}'\mathfrak{X}\mathfrak{X} \cdots \mathfrak{X}$. Then y is of class $\mathfrak{C}^{(k+1)}$, by Lemma 14.1 of Paper I.

In the case when the functions y of the space $\mathfrak{Y}^{(n)}$ are required to be of class $\mathfrak{C}^{(n)}$ uniformly $(\mathfrak{X}_0\mathfrak{P}'; \sigma)$, we must assume that the region $\mathfrak{X}_0$ is *convex*, in order to apply Taylor's theorem to the term R_3, and then the proof for completeness is practically the same as before.

Type D_0. Let $\mathfrak{X}^{(0)}$ be a measurable set of points in an m-dimensional euclidean space $\mathfrak{X}$. Then the class $\mathfrak{B}$ composed of all functions v on $\mathfrak{X}^{(0)}$ to $\mathfrak{R}$ (i.e., single-real-valued on $\mathfrak{X}^{(0)}$) which are bounded and measurable on $\mathfrak{X}^{(0)}$, is a complete linear metric space, if we set $\|v\|=$ the upper bound of $|v(x)|$.

Type D. $\mathfrak{Y}$ is the space of type B corresponding to a given range $\mathfrak{P}$ and a space $\mathfrak{B}$ of type D_0. This $\mathfrak{Y}$ may also be regarded as a class of functions on $\mathfrak{X}^{(0)}\mathfrak{P}$ to $\mathfrak{R}$, and we have

$$\|y\| = \text{upper bound of } \frac{|\,y(x,p)\,|}{|\,\sigma(p)\,|}$$

for those values of p for which $\sigma(p)\neq 0$ and for all x.

Type D_1. Let $\mathfrak{X}^{(1)}$ be a subset of the set $\mathfrak{X}^{(0)}$. Let $\mathfrak{Y}'$ be the class of all functions y of the space $\mathfrak{Y}$ (of type D, regarded as on $\mathfrak{X}^{(0)}\ \mathfrak{P}$ to $\mathfrak{R}$), which are continuous at the points of the set $\mathfrak{X}^{(1)}$ uniformly $(\mathfrak{P}; \sigma)$. The definition of $\|y\|$ is not changed. Then $\mathfrak{Y}'$ is a complete linear metric space.

II. Miscellaneous theorems on differentials and compact sets

3. In case $\mathfrak{Y}$ is a space of type A, i.e., a class of functions y on $\mathfrak{P}$ to $\mathfrak{B}$, and F is a function on a range $\mathfrak{P}'$ to $\mathfrak{Y}$, then if $F(p')=y$, we denote $y(p)$ also by $F(p'|p)$. Thus $F(p'|p)$ denotes an element of the space $\mathfrak{B}$, and a function G on $\mathfrak{P}'\mathfrak{P}$ to $\mathfrak{B}$ is determined by setting

$$F(p' \mid p) = G(p', p). \tag{3.1}$$

Conversely, if a function G on $\mathfrak{P}'\mathfrak{P}$ to $\mathfrak{B}$ has the right properties, the equation

(3.1) determines a corresponding function F on $\mathfrak{P}'$ to $\mathfrak{Y}$. We shall have frequent use for the notation $F(p'|p)$.

4. In this paragraph we consider a complete linear metric space $\mathfrak{Y}$ of type A_0, and a region $\mathfrak{X}_0$ of a linear metric space $\mathfrak{X}$. Consider also a function F on $\mathfrak{X}_0\mathfrak{P}'$ to $\mathfrak{Y}$, and the corresponding function G on $\mathfrak{X}_0\mathfrak{P}'\mathfrak{P}$ to $\mathfrak{B}$ determined by $F(x, p'|p)=G(x, p', p)$. Then we have

THEOREM I. *If the function F is of class $\mathfrak{C}^{(n)}$ on $\mathfrak{X}_0$ uniformly $([\mathfrak{X}_0]\mathfrak{P}'; \sigma')$, then the function G is of class $\mathfrak{C}^{(n)}$ on $\mathfrak{X}_0$ uniformly $([\mathfrak{X}_0]\mathfrak{P}'\mathfrak{P}; \sigma'\sigma)$, and conversely.*

The proof is made by direct application of the definition of the class $\mathfrak{C}'$, and induction.

5. **Compact sets and uniformity.*** In case a real-valued function F defined on a region $\mathfrak{X}_0$ of an m-dimensional euclidean space $\mathfrak{X}$ is of class $\mathfrak{C}'$ on $\mathfrak{X}_0$, then F is of class $\mathfrak{C}'$ on $\mathfrak{X}_0$ *uniformly* on every *bounded and closed* set $\mathfrak{X}^{(1)}$ contained in $\mathfrak{X}_0$. However, in this as in other uniformity theorems, the closure of the set $\mathfrak{X}^{(1)}$ is not the essential hypothesis.

In a metric space $\mathfrak{X}$, an infinite sequence of sets $\mathfrak{X}^{(m)}$ is said to have an *accumulation point* x_0 in case every neighborhood of x_0 contains points from an infinity of sets $\mathfrak{X}^{(m)}$. A set $\mathfrak{X}^{(1)}$ is *compact on* a set $\mathfrak{X}^{(0)}$ in case $\mathfrak{X}^{(1)}$ is included in $\mathfrak{X}^{(0)}$, and every infinite sequence of subsets of $\mathfrak{X}^{(1)}$ has an accumulation point in $\mathfrak{X}^{(0)}$.† This does not exclude the possibility that the sets $\mathfrak{X}^{(1)}$ and $\mathfrak{X}^{(0)}$ may contain only a finite number of points. A set $\mathfrak{X}^{(1)}$ compact on itself is called *self-compact*.

A *neighborhood* $(\mathfrak{X}^{(0)})_a$ of a set $\mathfrak{X}^{(0)}$ of points consists of the sum of the neighborhoods $(x_0)_a$ of all points x_0 of $\mathfrak{X}^{(0)}$.

LEMMA 5.1. *A set $\mathfrak{X}^{(1)}$, compact on $\mathfrak{X}$, is bounded.*‡

LEMMA 5.2. *A set $\mathfrak{X}^{(1)}$, which is compact on a region $\mathfrak{X}_0$, has a neighborhood $(\mathfrak{X}^{(1)})_a$ contained in $\mathfrak{X}_0$.*

These two lemmas are verified by the usual indirect proof.

LEMMA 5.3. *If $\mathfrak{X}$ is a composite space $(\mathfrak{U}, \mathfrak{B})$, and if the set $\mathfrak{X}^{(1)}$ is compact on the set $\mathfrak{X}^{(0)}$, then the projection $\mathfrak{U}^{(1)}$ of $\mathfrak{X}^{(1)}$ on the space $\mathfrak{U}$ is compact on the projection $\mathfrak{U}^{(0)}$ of $\mathfrak{X}^{(0)}$.*

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* The principal theorem of this sub-section is due to T. H. Hildebrandt.

† With the definition of compactness phrased in this way, it is possible to avoid any use of the postulate of Zermelo. Cf. Cipolla, Atti della Accademia Gioenia in Catania, vol. 6 (1913), Memoir V, *Sul postulato di Zermelo e la teoria dei limiti delle funzioni.*

‡ Cf. Fréchet, Rendiconti del Circolo Matematico di Palermo, vol. 22 (1906), p. 22.

Let $\mathfrak{X}^{(1)}$ be a set contained in the set $\mathfrak{X}^{(0)}$, and let F be a function on $\mathfrak{X}^{(0)}\mathfrak{P}$ to a metric space $\mathfrak{Y}$. Then we shall say that F is continuous on $\mathfrak{X}^{(0)}$ *uniformly* $(\mathfrak{X}^{(1)}\mathfrak{P};\sigma)$ in case F is continuous on $\mathfrak{X}^{(0)}$ uniformly $(\mathfrak{P};\sigma)$, and for every positive ϵ there exists a positive δ such that, for every x_1 in $\mathfrak{X}^{(1)}$ and every x_0 in $\mathfrak{X}^{(0)}$ satisfying $\|x_0, x_1\| < \delta$, and for every p, we have $\|F(x_0, p), F(x_1, p)\| \leqq \epsilon|\sigma(p)|$.

We say that a function F on $\mathfrak{X}^{(0)}\mathfrak{P}$ to a linear metric space $\mathfrak{Y}$ is *bounded* $(\mathfrak{P};\sigma)$ in case for each x of $\mathfrak{X}^{(0)}$ there exists a constant M_x such that $\|F(x, p)\| \leqq M_x|\sigma(p)|$ for every p. We obtain the definition of *modularity* as a special case of this in case a component of $\mathfrak{P}$ is a linear metric space $\mathfrak{Z}$, and $\|z\|$ is a factor of σ.

LEMMA 5.4. *Let the set $\mathfrak{X}^{(1)}$ be compact on the set $\mathfrak{X}^{(0)}$ of the metric space $\mathfrak{X}$, and let the function F on $\mathfrak{X}^{(0)}\mathfrak{P}$ to the metric space $\mathfrak{Y}$ be continuous on $\mathfrak{X}^{(0)}$ uniformly $(\mathfrak{P};\sigma)$. Then F is continuous on $\mathfrak{X}^{(0)}$ uniformly $(\mathfrak{X}^{(1)}\mathfrak{P};\sigma)$.*

LEMMA 5.5. *Let the set $\mathfrak{X}^{(1)}$ be compact on the set $\mathfrak{X}^{(0)}$ of the metric space $\mathfrak{X}$, and let $\mathfrak{Y}$ be a linear metric space. Let the function F on $\mathfrak{X}^{(0)}\mathfrak{P}$ to $\mathfrak{Y}$ be continuous on $\mathfrak{X}^{(0)}$ uniformly $(\mathfrak{P};\sigma)$, and bounded $(\mathfrak{P};\sigma)$. Then F is bounded $(\mathfrak{X}^{(1)}\mathfrak{P};\sigma)$.*

The usual indirect proofs suffice for both Lemma 5.4 and Lemma 5.5.

In case the spaces $\mathfrak{X}$ and $\mathfrak{Y}$ are linear, we say that the function F on $\mathfrak{X}_0\mathfrak{P}$ to $\mathfrak{Y}$ is of class $\mathfrak{C}'$ on $\mathfrak{X}_0$ *uniformly* $(\mathfrak{X}^{(1)}\mathfrak{P};\sigma)$ under the following conditions: (0) F is of class $\mathfrak{C}'$ on $\mathfrak{X}_0$ uniformly $(\mathfrak{P};\sigma)$; (1) dF is continuous on $\mathfrak{X}_0$ uniformly $(\mathfrak{X}^{(1)}\mathfrak{P}\mathfrak{X};\sigma\|dx\|)$; (2) dF is modular on $\mathfrak{X}$ uniformly $(\mathfrak{X}^{(1)}\mathfrak{P};\sigma)$; (3) for every positive ϵ there is a positive δ such that, whenever the points x_1 of $\mathfrak{X}^{(1)}$ and x_0 of $\mathfrak{X}_0$ satisfy $\|x_0-x_1\| < \delta$, then

$$\|F(x_0,p) - F(x_1,p) - dF(x_1,p;x_0-x_1)\| = \|R(x_0,x_1,p)\|\,\|x_0-x_1\| \leqq \epsilon\,|\sigma(p)|\,\|x_0-x_1\|.$$

This special uniformity is extended inductively in the usual manner to the class $\mathfrak{C}^{(n)}$.

THEOREM II. *Let the set $\mathfrak{X}^{(1)}$ be compact on the region $\mathfrak{X}_0$ of the linear metric space $\mathfrak{X}$, and suppose that the function F on $\mathfrak{X}_0\mathfrak{P}$ to the linear metric space $\mathfrak{Y}$ is of class $\mathfrak{C}^{(n)}$ on $\mathfrak{X}_0$ uniformly $(\mathfrak{P};\sigma)$. Then F is of class $\mathfrak{C}^{(n)}$ on $\mathfrak{X}_0$ uniformly $(\mathfrak{X}^{(1)}\mathfrak{P};\sigma)$.*

Consider first the case $n=1$. Properties (1) and (2) in the definition above follow from Lemmas 5.4 and 5.5. For property (3), we could proceed directly if the region $\mathfrak{X}_0$ were convex, but we shall use the indirect proof which covers

all cases. Suppose then that the last property does not hold. Then there exists a positive number ϵ such that for every positive integer m there exist a set $\mathfrak{X}^{(0m)}$ of points x_{0m} in $\mathfrak{X}_0$, a set $\mathfrak{X}^{(1m)}$ of points x_{1m} in $\mathfrak{X}^{(1)}$, and a set $\mathfrak{P}^{(m)}$ of elements p_m of the range $\mathfrak{P}$, such that

$$\|x_{0m} - x_{1m}\| \leqq \frac{1}{m}, \qquad \|R(x_{0m}, x_{1m}, p_m)\| > \epsilon \mid \sigma(p_m) \mid .$$

The sequence of sets $\{\mathfrak{X}^{(1m)}\}$ has an accumulation point x_0 which is in $\mathfrak{X}_0$, and hence there is a neighborhood $(x_0)_a$ contained in $\mathfrak{X}_0$. Then for every integer m' there is always an $m > m'$ such that there are points x_{0m} and x_{1m} of the sets $\mathfrak{X}^{(0m)}$ and $\mathfrak{X}^{(1m)}$ respectively, which are in $(x_0)_a$. The neighborhood $(x_0)_a$ is a convex region, so that we can apply Taylor's theorem* to obtain (omitting p)

$$R(x_{0m}, x_{1m})\|x_{0m} - x_{1m}\| = \int_0^1 \{dF(x_{1m} + (x_{0m} - x_{1m})r, x_{0m} - x_{1m}) - dF(x_{1m}, x_{0m} - x_{1m})\} dr.$$

We have already shown that dF is continuous on $\mathfrak{X}_0$ uniformly ($\mathfrak{X}^{(1)}\mathfrak{P}\mathfrak{X}$; $\sigma\|dx\|$), so that we have

$$\|R(x_{0m}, x_{1m})\| \leqq \epsilon \mid \sigma \mid$$

whenever $\|x_{0m} - x_{1m}\|$ is sufficiently small, i.e., when m' is sufficiently large. This is the desired contradiction.

The proof of the theorem is completed by an obvious induction.

In the case when the set $\mathfrak{X}^{(1)}$ contains only a finite number of points, the theorem is obvious and trivial.

6. **Equivalence of our definition with the classical definition of the class $\mathfrak{C}^{(n)}$.** Consider an m-dimensional euclidean space $\mathfrak{X}$, with the usual definition of distance. Let F be a single-real-valued function on a region $\mathfrak{X}_0$ of $\mathfrak{X}$ i.e., F is on $\mathfrak{X}_0$ to $\mathfrak{R}$. Then the function F is of class $\mathfrak{C}^{(n)}$ on $\mathfrak{X}_0$ according to Bolza† in case F has partial derivatives up to and including those of the nth order, all of which are continuous on $\mathfrak{X}_0$. The equivalence of this with our definition is stated in the following

THEOREM III. *If the function F is of class $\mathfrak{C}^{(n)}$ on $\mathfrak{X}_0$ according to the definition of Paper* I, *then F is of class $\mathfrak{C}^{(n)}$ on $\mathfrak{X}_0$ according to Bolza, and conversely.*

* See Paper II.

† See *Vorlesungen über Variationsrechnung*, p. 13.

It is convenient to introduce a general range $\mathfrak{P}$ with scale function σ, and to prove the more general

THEOREM III′. *If the function F on $\mathfrak{X}_0\mathfrak{P}$ to $\mathfrak{R}$ is of class $\mathfrak{C}^{(n)}$ on $\mathfrak{X}_0$ uniformly $(\mathfrak{P};\sigma)$, then F has partial derivatives up to and including those of the nth order, all of which are continuous on $\mathfrak{X}_0$ uniformly $(\mathfrak{P};\sigma)$, and bounded $(\mathfrak{P};\sigma)$ for each x of $\mathfrak{X}_0$. The converse is also true.*

Note that the function F itself is not necessarily bounded $(\mathfrak{P};\sigma)$. Although the range $\mathfrak{P}$ is essential in the induction, we can omit writing the variable p for the most part, as it enters "homogeneously."

We shall denote the coördinates of a point x by x^i $(i=1,\cdots,m)$, and the unit points on the various axes will be denoted by $u_i(i=1,\cdots,m)$, i.e., $u_i^j=0$ for $j\neq i$, and $=1$ for $j=i$.

Consider the theorem for $n=1$. The m functions $dF(x, u_i)$ are evidently the partial derivatives of F, and have the required properties.

To complete the proof by induction, suppose that F is of class $\mathfrak{C}^{(k+1)}$ on $\mathfrak{X}_0$. Then if the theorem is true for $n=k$, each function $dF(x, u_i)$ has partial derivatives of the kth order. Hence F has partial derivatives of the $(k+1)$st order.

To prove the converse, we define the function dF on $\mathfrak{X}_0\mathfrak{X}$ to $\mathfrak{R}$ by the equation

$$dF(x,dx) = \sum_1^m \frac{\partial F(x)}{\partial x^i}dx^i.$$

The continuity and linearity of dF are then obvious. For the third condition we have, by the classical theorem of the mean,

$$R(x_1,x)\|x_1 - x\| = \sum_1^m\left(\frac{\partial F(x+\theta(x_1-x))}{\partial x^i} - \frac{\partial F(x)}{\partial x^i}\right)(x_1^i - x^i) \qquad (0<\theta<1),$$

whence $R(x_1, x)$ approaches zero with $\|x_1-x\|$ uniformly $(\mathfrak{P};\sigma)$.

To complete the proof by induction, we assume that the theorem is true for $n=k$, and that F has derivatives of order $k+1$ with the required properties. Then it is readily verified that the function dF has partial derivatives up to the kth order which are continuous on $\mathfrak{X}_0$ uniformly $(\mathfrak{P}\mathfrak{X};\sigma\|dx\|)$ and bounded $(\mathfrak{P}\mathfrak{X};\sigma\|dx\|)$. Hence dF is of class $\mathfrak{C}^{(k)}$ on $\mathfrak{X}_0$ uniformly $(\mathfrak{P}\mathfrak{X};\sigma\|dx\|)$, so that F is of class $\mathfrak{C}^{(k+1)}$ as required.

7. The lemmas of this paragraph will occasionally be of use in dealing with differentials of order higher than the first. All the spaces $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ considered are assumed to be linear metric spaces.

There will be occasion to consider differentiation of functions $F(p, x)$ on $\mathfrak{P}\mathfrak{X}_p$ to $\mathfrak{Y}$, where the range $\mathfrak{X}_p$ of the argument x depends on the argument p. In such cases it is obviously suitable to write the range $\mathfrak{P}$ first. The range $\mathfrak{P}$ may of course be composite, and $\mathfrak{X}_p$ may then actually depend on only one component of $\mathfrak{P}$. We say that a function F on $\mathfrak{P}\mathfrak{X}_p$ to $\mathfrak{Y}$ is continuous on $\mathfrak{X}_p$ uniformly $(\mathfrak{P}\mathfrak{X}_p; \sigma)$ in case for every positive ϵ there is a positive δ such that, for every p and every x_1 and x_2 in $\mathfrak{X}_p$ having $\|x_1 - x_2\| \leqq \delta$ we have

$$\|F(p, x_1) - F(p, x_2)\| \leqq \epsilon \,|\, \sigma(p) \,| \,.$$

This indicates how we define the class $\mathfrak{C}'$ on $\mathfrak{X}_p$ uniformly $(\mathfrak{P}\mathfrak{X}_p; \sigma)$.

LEMMA 7.1. *Let the function F on $\mathfrak{P}\mathfrak{X}_p\mathfrak{Y}$ to $\mathfrak{Z}$ be distributive on $\mathfrak{Y}$ for each p and x. Let b be a positive constant, and let $\mathfrak{Y}^{(b)}$ denote the set of points y such that $\|y\| = b$. Then if F is of class $\mathfrak{C}'$ on $\mathfrak{X}_p$ uniformly $(\mathfrak{P}\mathfrak{X}_p\mathfrak{Y}^{(b)}; \sigma)$, F is also of class $\mathfrak{C}'$ on $\mathfrak{X}_p$ uniformly $(\mathfrak{P}\mathfrak{X}_p\mathfrak{Y}; \sigma\|y\|)$, and conversely. Moreover the differential d_xF is linear on $\mathfrak{Y}$ uniformly $(\mathfrak{P}\mathfrak{X}_p\mathfrak{X}; \sigma\|dx\|)$.*

The lemma can readily be extended by induction to the case where F is of class $\mathfrak{C}^{(n)}$, and there are m spaces entering after the fashion of the space $\mathfrak{Y}$. But we shall not need anything more general than the lemma stated. The proof is as follows.

We first define a function G on $\mathfrak{P}\mathfrak{X}_p\mathfrak{Y}\mathfrak{X}$ to $\mathfrak{Z}$ by the equations

$$G(p, x, y\,; dx) = d_xF(p, x, yb'; dx)\frac{1}{b'},$$

$$b' = \frac{b}{\|y\|} \qquad (y \neq y_*),$$

$$G(p, x, y_*\,; dx) = z_*.$$

Then it is easy to verify that G is the differential d_xF of F, and satisfies the conditions for F to be of class $\mathfrak{C}'$ on $\mathfrak{X}_p$ uniformly $(\mathfrak{P}\mathfrak{X}_p\mathfrak{Y}; \sigma\|y\|)$. The converse part of the lemma is obvious.

By use of Lemma 11.1 of Paper I, we can show that d_xF is distributive on $\mathfrak{Y}$. The modularity of d_xF on $\mathfrak{Y}$ uniformly $(\mathfrak{P}\mathfrak{X}_p\mathfrak{X}; \sigma\|dx\|)$ is equivalent to the modularity of d_xF on $\mathfrak{X}$ uniformly $(\mathfrak{P}\mathfrak{X}_p\mathfrak{Y}; \sigma\|y\|)$.

LEMMA 7.2. *Let the function F on $\mathfrak{P}\mathfrak{X}_p\mathfrak{Y}$ to $\mathfrak{Z}$ be linear on $\mathfrak{Y}$ uniformly $(\mathfrak{P}\mathfrak{X}_p; \sigma)$, and let b be a positive number. Then if F is of class $\mathfrak{C}^{(n)}$ on $\mathfrak{X}_p$ uniformly $(\mathfrak{P}\mathfrak{X}_p\mathfrak{Y}; \sigma\|y\|)$, F is also of class $\mathfrak{C}^{(n)}$ on the composite region $(\mathfrak{X}_p(y_*)b)$ uniformly $(\mathfrak{P}\mathfrak{X}_p(y_*)_b; \sigma)$, and conversely.*

This lemma could also be extended to the case where there are m spaces entering in the fashion of the space $\mathfrak{Y}$.

To prove the lemma as it stands, we consider first the case $n=1$. Denote the region $(\mathfrak{X}_p,(y_*)_b)$ by $\mathfrak{W}_p$. The differential d_xF, and the differential d_yF defined by

$$d_yF(p,x,y;dy) = F(p,x,dy), \tag{7.1}$$

evidently satisfy the uniformity requirements of Lemma 13.2 of Paper I, so that we can say that F is of class $\mathfrak{C}'$ on $\mathfrak{W}_p$ uniformly $(\mathfrak{P}\mathfrak{W}_p;\sigma)$.

To complete the proof by induction, we assume that the lemma is true for $n=k$, and that F is of class $\mathfrak{C}^{(k+1)}$ on $\mathfrak{X}_p$ uniformly $(\mathfrak{P}\mathfrak{X}_p\mathfrak{Y};\sigma\|y\|)$. Then d_xF is of class $\mathfrak{C}^{(k)}$ on $\mathfrak{X}_p$ uniformly $(\mathfrak{P}\mathfrak{X}_p\mathfrak{Y}\mathfrak{X};\sigma\|y\|\ \|dx\|)$. Also d_xF is linear on $\mathfrak{Y}$ uniformly $(\mathfrak{P}\mathfrak{X}_p\mathfrak{X};\sigma\|dx\|)$. Hence by the present lemma for $n=k$, d_xF is of class $\mathfrak{C}^{(k)}$ on $\mathfrak{W}_p$ uniformly $(\mathfrak{P}\mathfrak{W}_p\mathfrak{X};\sigma\|dx\|)$. By Lemma 14.1 of Paper I, d_yF (as defined by equation (7.1)), is of class $\mathfrak{C}^{(k)}$ on $\mathfrak{X}_p$ uniformly $(\mathfrak{P}\mathfrak{X}_p\mathfrak{Y};\sigma\|dy\|)$, and since d_yF is independent of y, it is also of class $\mathfrak{C}^{(k)}$ on $\mathfrak{W}_p$ uniformly $(\mathfrak{P}\mathfrak{W}_p\mathfrak{Y};\sigma\|dy\|)$. Hence the sum $d_xF+d_yF=d_wF$ is of class $\mathfrak{C}^{(k)}$ on $\mathfrak{W}_p$ uniformly $(\mathfrak{P}\mathfrak{W}_p\mathfrak{W};\sigma\|dw\|)$, so that F is of class $\mathfrak{C}^{(k+1)}$ on $\mathfrak{W}_p$ uniformly $(\mathfrak{P}\mathfrak{W}_p;\sigma)$. This completes the induction.

To prove the converse for $n=1$, we have from Lemma 13.2 of Paper I that F is of class $\mathfrak{C}'$ on $\mathfrak{X}_p$ uniformly $(\mathfrak{P}\mathfrak{X}_p(y_*)_b;\ \sigma)$, and hence uniformly $(\mathfrak{P}\mathfrak{X}_p\mathfrak{Y}^{(b')};\sigma)$, where $b'<b$. Then Lemma 7.1 gives the desired conclusion.

To complete the proof by induction, we make the usual assumptions. Then d_xF is of class $\mathfrak{C}^{(k)}$ on $\mathfrak{W}_p$ uniformly $(\mathfrak{P}\mathfrak{W}_p\mathfrak{X};\sigma\|dx\|)$, and linear on $\mathfrak{Y}$ uniformly $(\mathfrak{P}\mathfrak{X}_p\mathfrak{X};\sigma\|dx\|)$. Hence by the lemma for $n=k$, d_xF is of class $\mathfrak{C}^{(k)}$ on $\mathfrak{X}_p$ uniformly $(\mathfrak{P}\mathfrak{X}_p\mathfrak{Y}\mathfrak{X};\sigma\|y\|\ \|dx\|)$, so that by definition, F is of class $\mathfrak{C}^{(k+1)}$ on $\mathfrak{X}_p$ uniformly $(\mathfrak{P}\mathfrak{X}_p\mathfrak{Y};\sigma\|y\|)$.

III. The solution of the equation $G(x, y)=y_*$ on subspaces

8. **Nature of the subspaces considered.** We shall be considering linear metric subspaces $\mathfrak{X}'$ of linear metric spaces $\mathfrak{X}$, the definition of the operations $\|\ \|$, $\oplus$, $\odot$ being unchanged in the subspace. As a special case we may have $\mathfrak{X}'=\mathfrak{X}$. In particular, following out the practice of Parts IV and V of Paper I, we shall denote by $\mathfrak{Y}'$ a complete linear metric subspace of a complete linear metric space $\mathfrak{Y}$. If we set $\mathfrak{W}=(\mathfrak{X},\ \mathfrak{Y})$, $\mathfrak{W}\ =(\mathfrak{X}',\ \mathfrak{Y}')$, then $\mathfrak{W}'$ is a linear metric subspace of $\mathfrak{W}$.

Corresponding to a region $\mathfrak{X}_0$ of $\mathfrak{X}$ we shall denote by $(\mathfrak{X}_0)'$ the set of points common to $\mathfrak{X}_0$ and a subspace $\mathfrak{X}'$. If F is a function on $\mathfrak{X}_0$ to $\mathfrak{Y}$ such that, when x is in $(\mathfrak{X}_0)'$, $F(x)$ is in the subspace $\mathfrak{Y}'$ of $\mathfrak{Y}$, we shall say that F *as on* $(\mathfrak{X}_0)'$ *is to* $\mathfrak{Y}'$. We propose to determine the nature of the solution of the equation $G(x,\ y)=y_*$, when the initial solution $(x_0,\ y_0)$ is in $\mathfrak{W}'$, and the function G as on $(\mathfrak{W}_0)'$ is to $\mathfrak{Y}'$.

9. **Limits, regions, and differentials in subspaces.** We prove the following lemmas.

LEMMA 9.1. *If a sequence* $\{y_n'\}$ *of elements of* $\mathfrak{Y}'$ *has a limit* y *in* $\mathfrak{Y}$, *then* y *is in* $\mathfrak{Y}'$.

This follows from the completeness of $\mathfrak{Y}'$, and the uniqueness of limit in $\mathfrak{Y}$.

LEMMA 9.2. *If* $\mathfrak{X}_0$ *is a region of* $\mathfrak{X}$ *containing points of* $\mathfrak{X}'$, *then the set* $(\mathfrak{X}_0)'$ *is a region in the space* $\mathfrak{X}'$.

This follows readily from the definition of region.

LEMMA 9.3. *Let* F *be a function on* $\mathfrak{X}_0$ *to* $\mathfrak{Y}$, *having a differential* dF *at a point* x_0' *of the region* $(\mathfrak{X}_0)'$, *and suppose that* F *as on* $(\mathfrak{X}_0)'$ *is to* $\mathfrak{Y}'$. *Then the differential* dF *as on* $\mathfrak{X}'$ *is to* $\mathfrak{Y}'$.

This follows from Lemma 11.1 of Paper I, and Lemma 9.1.

10. The theorems of this paragraph are corollaries or additions to Theorems IV and V of Paper I. The latter, we shall call Theorems IV^{I}, V^{I} respectively. For convenience we retain here all the notations, hypotheses, and conclusions of Theorems IV^{I} and V^{I}, and state only the additional hypotheses and conclusions.

THEOREM IV. *Suppose that the initial solution* (x_0, y_0) *is in* $\mathfrak{W}'$, *and that the function* G *as on* $(\mathfrak{W}_0)'$ *is to* $\mathfrak{Y}'$. *Suppose also that the reciprocal function* L_0 *as on* $\mathfrak{Y}'$ *is to* $\mathfrak{Y}'$. *Then the solution* Y *as on* $((x_0)_b)'$ *is to* $((y_0)_a)'$, *and the reciprocal* L *of* $d_yG(x, Y(x); dy)$ *as on* $((x_0)_b)'\mathfrak{Y}'$ *is to* $\mathfrak{Y}'$.

By Lemma 9.3, the function G, as on $(\mathfrak{W}_0)'$ to $\mathfrak{Y}'$, is of class $\mathfrak{C}'$ on $(\mathfrak{W}_0)'$. Thus all the hypotheses of Theorem IV^{I} (for $n=1$, at least) are satisfied with $\mathfrak{X}$ replaced by $\mathfrak{X}'$ and $\mathfrak{Y}$ by $\mathfrak{Y}'$, so that there exist positive constants a' and b' and a unique function Y' on $((x_0)_{b'})'$ to $((y_0)_{a'})'$ satisfying the conclusions C_1 to C_4 of Theorem IV^{I}. By inspection of the proofs of Theorem IV^{I} and Lemma 16.2^{I}, it is seen that the constants a' and b' are restricted only by the continuity and modularity properties of G, d_yG, and L_0, so that we may take $a'=a$, $b'=b$. Then by the uniqueness of the solution, we have $Y'=Y$ on $((x_0)_b)'$.

THEOREM V. *In addition to the assumptions of Theorem* IV, *suppose that the subspace* $\mathfrak{X}'$ *is identical with the space* $\mathfrak{X}$. *Then the sheet* $\mathfrak{W}^{(0)}$ *of solutions, defined in Theorem* V^{I}, *lies wholly in the space* $\mathfrak{W}'$.

To prove this statement, let w_1 be a point of the sheet $\mathfrak{W}^{(0)}$, and let w_1 be connected to the initial point $w_0=(x_0, y_0)$ by a continuous function W on the interval (01) to $\mathfrak{W}^{(0)}$. Then by the definition of a sheet, and Theorem IV, there exist points r_2 in the interval (01) such that, for $0 \leqq r \leqq r_2$ we have (1) $W(r)$ is in $\mathfrak{W}'$, and (2) the reciprocal L of d_yG at $W(r)$ has the property that L as on $\mathfrak{Y}'$ is to $\mathfrak{Y}'$. Let r_3 be the upper bound of such points r_2. Then by the continuity of the function W and of the reciprocal of d_yG, and by Lemma 9.1, the conditions (1) and (2) are satisfied for $r=r_3$. If $r_3 \neq 1$, there is by Theorem IV a neighborhood of r_3 on which the same conditions hold. This is a contradiction, so that the point w_1 is actually in $\mathfrak{W}'$.

11. **Extent of the maximal sheet of solutions in a subspace.** When we consider the equation $G(x, y)=y_*$ in a subspace $\mathfrak{Y}'$ of $\mathfrak{Y}$, the maximal sheet of solutions may extend farther in the subspace than in the main space. This is shown by such a simple example as the following. Let the space $\mathfrak{X}$ be a one-dimensional space, and the space $\mathfrak{Y}$ a two-dimensional space, and denote the coördinate of a point x simply by x, and the coördinates of a point y by y_1 and y_2. Let $\mathfrak{Y}'$ be the one-dimensional subspace of $\mathfrak{Y}$ for which $y_1=y_2$. We take for the function G the one whose coördinates are

$$(3 - x)y_1 + (x - 1)y_2 - x,$$

$$(1 + x)y_1 + (1 - x)y_2 - x.$$

Then G as on $\mathfrak{X}\mathfrak{Y}'$ is to $\mathfrak{Y}'$. Also for each $x \neq 1$, the differential d_yG has a reciprocal K on $\mathfrak{Y}$ to $\mathfrak{Y}$, and K as on $\mathfrak{Y}'$ is to $\mathfrak{Y}'$. Now a solution of the equation $G=y_*$ has the form $y_1=y_2=x/2$, valid for all values of x, and unique except for $x=1$. But for $x=1$, d_yG has no reciprocal in the space $\mathfrak{Y}$, so that a maximal sheet $\mathfrak{W}^{(0)}$ of solutions in the space $\mathfrak{Y}$ would have the point $x=1$, $y_1=y_2=1/2$, for a boundary point. But even at $x=1$, d_yG has a reciprocal in the space $\mathfrak{Y}'$, so that the maximal sheet $\mathfrak{W}^{(0)}$ in the space $\mathfrak{Y}'$ can have no boundary point.

IV. A special mode of dependence of the function G upon x and y, and an extended implicit function theorem

12. **Nature of the spaces $\mathfrak{X}$ and $\mathfrak{Y}$, and special notations.** In this section we shall be considering spaces $\mathfrak{X}$ and $\mathfrak{Y}$ of type B. More specifically let $\mathfrak{U}$ be a linear metric space and $\mathfrak{V}$ be a complete linear metric space. Let $\mathfrak{P}$ be a general range, and let the scale function σ be identically equal to unity. Then let $\mathfrak{X}$ be the space of type B consisting of functions x on $\mathfrak{P}$ to $\mathfrak{U}$, and let $\mathfrak{Y}$ be the space of type B consisting of functions y on $\mathfrak{P}$ to $\mathfrak{V}$. The space $\mathfrak{Y}$ therefore is complete. It is not necessary to restrict the spaces $\mathfrak{X}$ and $\mathfrak{Y}$ so strongly for the first lemmas, but the possible extensions are easily noted.

Corresponding to a region $\mathfrak{X}_0$ of a space $\mathfrak{X}$, there will be a system of regions $\mathfrak{U}_p$ in the space $\mathfrak{U}$, where $\mathfrak{U}_p$ consists of all those points u for which there is a function x in $\mathfrak{X}_0$ having $x(p)=u$. It is readily verified that each $\mathfrak{U}_p$ so defined actually is a region. Conversely, corresponding to an arbitrary system of regions $\mathfrak{U}_p$ there is usually a region $\mathfrak{X}_0$ to which the system of regions $\mathfrak{U}_p$ corresponds as above. But such a region $\mathfrak{X}_0$ will not always be uniquely determined by this condition. If a region $\mathfrak{X}_0$ exists corresponding to a system of regions $\mathfrak{U}_p$, the most inclusive such region consists of all points x of the space $\mathfrak{X}$ for which there is a positive constant a_x such that the neighborhood $(x(p))_{a_x}$ is in $\mathfrak{U}_p$, for every p. To avoid questions of existence we shall throughout this section assume that the region $\mathfrak{X}_0$ is given and that the system of regions $\mathfrak{U}_p$ corresponds to it. We shall also retain the notation $F(p'|p)$ introduced in § 3.

13. **Preliminary theorems on differentials and reciprocals.** We prove the following lemmas.

LEMMA 13.1. *Let G be a function on $\mathfrak{X}_0\mathfrak{P}'$ to $\mathfrak{Y}$, of class $\mathfrak{C}^{(n)}$ on $\mathfrak{X}_0$ uniformly $(\mathfrak{P}'; \sigma')$, and suppose that $G(x, p'|p)$ depends on the function x only at p, for every p and p'. Then the function H on $\mathfrak{P}\mathfrak{U}_p\mathfrak{P}'$ to $\mathfrak{Y}$ defined by*

$$G(x,p' \mid p) = H(p,u,p'), \quad u = x(p),$$

is of class $\mathfrak{C}^{(n)}$ on each region $\mathfrak{U}_p$ uniformly $(\mathfrak{P}'; \sigma')$.

The ranges of uniformity in this lemma may be modified in special cases. E.g., *if the region $\mathfrak{U}_p$ is the same for every p, then $\mathfrak{P}$ may be inserted in the range of uniformity in the conclusion. If the region $\mathfrak{X}_0$ is a neighborhood $(x_0)_a$, then $\mathfrak{X}_0$ and $\mathfrak{P}\mathfrak{U}_p$ may be inserted in the ranges of uniformity of the hypothesis and conclusion respectively.* The outline of the proof is unaltered.

To prove the lemma we must first show that the differential d_xG has the property that $d_xG(x, p'; dx|p)$ depends on x and dx only at p. Let $x_1(p)=x_2(p)$, $d_1x(p)=d_2x(p)$, for a certain p. Then from the assumed property of the function G, Lemma 11.1 of Paper I, and the uniqueness of limits, we find that $d_xG(x_1, p'; d_1x|p)=d_xG(x_2, p'; d_2x|p)$. Hence we can define the function d_uH on $\mathfrak{P}\mathfrak{U}_p\mathfrak{P}'\mathfrak{U}$ to $\mathfrak{Y}$ by

$$d_xG(x,p';dx \mid p) = d_uH(p,u,p';du), \quad u = x(p), \quad du = dx(p).$$

To prove the continuity of d_uH, let

$$\begin{aligned} u_2 &= x_2(p_0), \\ x_1(p) &= x_2(p) + u_1 - u_2 \text{ for every } p, \\ dx(p) &= du \text{ for every } p. \end{aligned}$$

We have immediately

$$\|u_1 - u_2\| = \|x_1 - x_2\|, \quad \|du\| = \|dx\|.$$

If u_2 is in $\mathfrak{U}_{p_0}$, x_2 can be assigned in $\mathfrak{X}_0$, and x_1 is then also in $\mathfrak{X}_0$ if u_1 is sufficiently near u_2. Then

$$d_uH(u_1,p_0,p';du) - d_uH(u_2,p_0,p';du)$$
$$= d_xG(x_1,p';dx \mid p_0) - d_xG(x_2,p';dx \mid p_0).$$

From this the required continuity of d_uH is apparent.

The other two conditions for H to be of class $\mathfrak{C}'$ are obtained as readily.

We complete the proof by induction. Assume that the lemma is true for $n=k$, and that G is of class $\mathfrak{C}^{(k+1)}$. Then d_xG is of class $\mathfrak{C}^{(k)}$ on $\mathfrak{X}_0$ uniformly $(\mathfrak{P}'\mathfrak{X};\sigma'\|dx\|)$, and $d_xG(x,p';dx|p)$ depends on x only at p. If we set $dx(p)=du$ for all p, then the conditions of the lemma are satisfied with G replaced by d_xG, $\mathfrak{P}'$ replaced by $\mathfrak{P}'\mathfrak{U}$, and $n=k$. Hence d_uH is of class $\mathfrak{C}^{(k)}$ on each $\mathfrak{U}_p$ uniformly $(\mathfrak{P}'\mathfrak{U};\sigma'\|du\|)$, so that H is of class $\mathfrak{C}^{(k+1)}$ as required.

The next lemma is a converse of the preceding, but it requires a stronger uniformity.

LEMMA 13.2. *Consider a region $\mathfrak{X}_0$ of the space $\mathfrak{X}$, and the corresponding system of regions $\mathfrak{U}_p$ of the space $\mathfrak{U}$. Let H be a function on $\mathfrak{P}\mathfrak{U}_p\mathfrak{P}'$ to $\mathfrak{B}$, of class $\mathfrak{C}^{(n)}$ on $\mathfrak{U}_p$ uniformly $(\mathfrak{P}\mathfrak{U}_p\mathfrak{P}';\sigma')$, and such that $\|H(p,x(p),p')\|$ is bounded on $\mathfrak{P}$ for every x in $\mathfrak{X}_0$ and every p' of $\mathfrak{P}'$. Then the function G on $\mathfrak{X}_0\mathfrak{P}'$ to $\mathfrak{Y}$ defined by*

$$G(x,p' \mid p) = H(p,x(p),p')$$

is of class $\mathfrak{C}^{(n)}$ on $\mathfrak{X}_0$ uniformly $(\mathfrak{X}_0\mathfrak{P}';\sigma')$.

Take first the case $n=1$, and define the function d_xG on $\mathfrak{X}_0\mathfrak{P}'\mathfrak{X}$ to $\mathfrak{Y}$ by the equation

$$d_xG(x,p';dx \mid p) = d_uH(p,x(p),p';dx(p)).$$

That d_xG is *to* $\mathfrak{Y}$ follows from the uniform modularity of the function d_uH. The conditions for G to be of class $\mathfrak{C}'$ with d_xG as its differential are readily verified.

To complete the proof by induction, we need to use Lemma 7.2. Assume that the present lemma is true for $n=k$, and that H is of class $\mathfrak{C}^{(k+1)}$ on $\mathfrak{U}_p$ uniformly $(\mathfrak{P}\mathfrak{U}_p\mathfrak{P}';\sigma')$. Then d_uH is of class $\mathfrak{C}^{(k)}$ on $\mathfrak{U}_p$ uniformly $(\mathfrak{P}\mathfrak{U}_p\mathfrak{P}'\mathfrak{U};$ $\sigma'\|du\|)$, and hence of class $\mathfrak{C}^{(k)}$ on $\mathfrak{U}_p(u_*)_a$ uniformly $(\mathfrak{P}\mathfrak{U}_p\mathfrak{P}'(u_*)_a;\sigma')$, where a is a positive number, by Lemma 7.2. Then by the present lemma for

$n=k$, $d_z G$ is of class $\mathfrak{C}^{(k)}$ on $\mathfrak{X}_0(x_*)_a$ uniformly $(\mathfrak{X}_0\mathfrak{P}'(x_*)_a;\sigma')$. Application of Lemma 7.2 in the reverse direction now gives the desired conclusion.

The next lemma is concerned with reciprocal linear functions. We have a correspondence between functions K_1 on $\mathfrak{Y}$ to $\mathfrak{Y}$ which are linear on $\mathfrak{Y}$ and such that $K_1(y|p)$ depends on y only at p, and functions K_2 on $\mathfrak{V}\mathfrak{P}$ to $\mathfrak{V}$ which are linear on $\mathfrak{V}$ uniformly on $\mathfrak{P}$, by means of the equation

$$K_1(y \mid p) = K_2(y(p), p).$$

It is evident that the function K_1 corresponding to such a function K_2 is actually linear on $\mathfrak{Y}$. In terms of this correspondence the lemma can be stated as follows.

LEMMA 13.3. *Let the function K_1 on $\mathfrak{Y}$ to $\mathfrak{Y}$ be linear on $\mathfrak{Y}$, and such that $K_1(y|p)$ depends on y only at p, and suppose that K_1 has a reciprocal L_1. Then L_1 necessarily has the same property as K_1. Moreover, the functions K_2 and L_2 on $\mathfrak{V}\mathfrak{P}$ to $\mathfrak{V}$ corresponding to K_1 and L_1 respectively, are reciprocal for every p of $\mathfrak{P}$. Conversely, if K_2 and L_2 on $\mathfrak{V}\mathfrak{P}$ to $\mathfrak{V}$ are reciprocal for every p, and linear on $\mathfrak{V}$ uniformly on $\mathfrak{P}$, then the corresponding functions K_1 and L_1 are reciprocal. Finally, the moduli of K_1 and K_2 are equal, and the moduli of L_1 and L_2 are equal.*

We first show that $K_1(y|p_0)=v_*$ if and only if $y(p_0)=v_*$. If $y(p_0)=v_*$, then $K_1(y|p_0)=K_1(y_*|p_0)=v_*$. If $K(y|p_0)=v_*$, define a function y_1 as follows: $y_1(p_0)=y(p_0)$, $y_1(p)=v_*$ for $p\neq p_0$. Then $K_1(y_1|p_0)=K_1(y|p_0)=v_*$, and hence $K_1(y_1)=y_*$. Since K_1 has a reciprocal L_1, $y_1=y_*$, whence $y(p_0)=v_*$.

From this we can show that $L_1(y|p)$ depends on y only at p. For, let $y_1(p)=y_2(p)$. Then

$$K_1(L_1(y_1 - y_2) \mid p) = y_1(p) - y_2(p) = v_*,$$
$$L_1(y_1 - y_2 \mid p) = v_*.$$

The reciprocal properties in the first part of the lemma and in the converse are practically obvious.

To show the equality of the moduli, we have, for K_1 and K_2,

$$\begin{aligned}\|K_1(y)\| &= \overline{B}\|K_1(y \mid p)\| = \overline{B}\|K_2(y(p), p)\| \\ &\leqq M(K_2)\overline{B}\|y(p)\| = M(K_2)\|y\|,\end{aligned}$$
$$\|K_2(v,p)\| = \|K_1(y_1 \mid p)\| \leqq \|K_1(y_1)\| \leqq M(K_1)\|v\|,$$

where $\overline{B}$ denotes the upper bound as to p, and $y_1(p)=v$ for every p.

14. **Solution of $H(x(p), y(p), p)=v_*$.** If we set $\mathfrak{Z}=(\mathfrak{U}, \mathfrak{V})$, $\mathfrak{W}=(\mathfrak{X}, \mathfrak{Y})$, then $\mathfrak{Z}$ and $\mathfrak{W}$ are related in the same way as $\mathfrak{U}$ and $\mathfrak{X}$. Also there corresponds

to a region $\mathfrak{W}_0$ a system of regions $\mathfrak{Z}_p$ as defined in the opening paragraphs of this section. The following theorem is a corollary of Theorem IV of Paper I and will serve as a basis for the extended implicit function theorem of the next subsection. The principal point of the theorem is that there exist constants a and b independent of p.

THEOREM VI. *Consider a region $\mathfrak{W}_0$, and the corresponding system of regions $\mathfrak{Z}_p$, a point $w_0=(x_0, y_0)$ of $\mathfrak{W}_0$, and a function H on $\mathfrak{P}\mathfrak{Z}_p$ to $\mathfrak{V}$ with the following properties*:

(H_1) *$H(p, x_0(p), y_0(p)) = v_* $ for every p*;

(H_2) *H is of class $\mathfrak{C}^{(n)}$ on $\mathfrak{Z}_p$ uniformly on $\mathfrak{P}\mathfrak{Z}_p$*;

(H_3) *$d_v H(p, x_0(p), y_0(p); dv) \equiv K_0(dv, p)$ has a reciprocal L_0, and L_0 is linear on $\mathfrak{V}$ uniformly on $\mathfrak{P}$.*

Under these hypotheses there exist positive constants a and b and a function $\mathfrak{V}$ on $\mathfrak{P}$ $(x_0(p))_b$ to $\mathfrak{V}$ with the following properties:

(C_1) *the region $((x_0(p))_b, (y_0(p))_a$ is contained in $\mathfrak{Z}_p$ for every p*;

(C_2) *for every p and every u in $(x_0(p))_b$, $V(p, u)$ is the unique solution of the equation*

$$H(p,u,v) = v_*,$$

having v in $(y_0(p))_a$;

(C_3) *the differential $d_v H(p, u, V(p, u); dv)$ has a reciprocal L on $\mathfrak{V}\mathfrak{P}(x_0(p))_b$ to $\mathfrak{V}$, which is linear on $\mathfrak{V}$ uniformly on $\mathfrak{P}(x_0(p))_b$*;

(C_4) *the function V is of class $\mathfrak{C}^{(n)}$ on $(x_0(p))_b$ uniformly on $\mathfrak{P}(x_0(p))_b$.*

We reduce this theorem to Theorem IV of Paper I by considering the function G on $\mathfrak{W}_0$ defined by

$$G(x,y \mid p) - H(p,x(p),y(p)).$$

If the functional values of G are to be in the space $\mathfrak{Y}$, $\|H(p, x(p), y(p))\|$ must be bounded on $\mathfrak{P}$. This will be so if the point (x, y) lies in a neighborhood $((x_0, y_0))_c$ which in turn lies in the region $\mathfrak{W}_0$. To prove this we apply the hypotheses (H_1) and (H_2) and Taylor's theorem.*

After we replace the region $\mathfrak{W}_0$ by the neighborhood $(w_0)_c$, we can readily show by Lemmas 13.2 and 13.3 that the point w_0 and the function G on $(w_0)_c$ to $\mathfrak{Y}$ satisfy all the hypotheses of Theorem IV of Paper I.

The next step is to show that the function Y obtained from Theorem IV[I] is such that $Y(x|p)$ depends on x only at p. Suppose that for a certain p_0 we have

$$x_1(p_0) = x_2(p_0), \quad Y(x_1 \mid p_0) \neq Y(x_2 \mid p_0).$$

* See Paper II.

Then a second solution $\bar{Y}(x_1)$, in $(y_0)_a$, but not equal to $Y(x_1)$, is given by

$$\bar{Y}(x_1 \mid p_0) = Y(x_2 \mid p_0), \quad \bar{Y}(x_1 \mid p) = Y(x_1 \mid p) \quad (p \neq p_0).$$

And this contradicts the uniqueness of the solution.

If we set $V(p, u) = Y(x|p)$, where $x(p) = u$, then the function V and the constants a and b obviously satisfy the conditions (C_1) and (C_2). The conclusion (C_3) follows from Lemma 13.3, and (C_4) from Lemma 13.1 and the remark appended thereto.

15. **An extended implicit function theorem.** The theorem we shall obtain is a generalization of those given by Bolza and others, but our method of proof is entirely different. The theorem is "extended" in the sense that there is a set of initial solutions instead of only a single initial solution, and that there is a uniform neighborhood of the initial set in which there is a unique solution.

If $\mathfrak{U}^{(0)}$ is a set of points in the linear metric space $\mathfrak{U}$, we shall denote such a uniform neighborhood by $(\mathfrak{U}^{(0)})_a$, as in § 5.

We shall be considering a set $\mathfrak{Z}^{(0)}$ in the composite space $\mathfrak{Z} = (\mathfrak{U}, \mathfrak{V})$, having the property that if (u_1, v_1), (u_2, v_2) are distinct points of $\mathfrak{Z}^{(0)}$, then $u_1 \neq u_2$. Let $\mathfrak{U}^{(0)}$ be the "projection" of $\mathfrak{Z}^{(0)}$ on the space $\mathfrak{U}$. Then such a set $\mathfrak{Z}^{(0)}$ defines a single-valued function $V^{(0)}$ on $\mathfrak{U}^{(0)}$ to $\mathfrak{V}$.

THEOREM VII. *Let $\mathfrak{Z}^{(0)}$ be a set of points defining a single-valued function $V^{(0)}$ on $\mathfrak{U}^{(0)}$, as just described, and let G be a function on a region $\mathfrak{Z}_0$ to $\mathfrak{V}$, with the following properties:*

(H_1) $\mathfrak{Z}^{(0)}$ is in $\mathfrak{Z}_0$;

(H_2) $G(z) = G(u, v) = v_*$ on $\mathfrak{Z}^{(0)}$;

(H_3) G is of class $\mathfrak{C}^{(n)}$ on $\mathfrak{Z}_0$ uniformly on $\mathfrak{Z}_0$;

(H_4) $d_vG(z; dv)$ has a reciprocal $L(z; dv)$ for every z in $\mathfrak{Z}^{(0)}$;

(H_5) the set $\mathfrak{Z}^{(0)}$ is bounded;

(H_6) there is a neighborhood $(\mathfrak{Z}^{(0)})_c$ in $\mathfrak{Z}_0$;

(H_7) the function $V^{(0)}$ is continuous on $\mathfrak{U}^{(0)}$ uniformly on $\mathfrak{U}^{(0)}$;

(H_8) $L(z; dv)$ is linear on $\mathfrak{V}$ uniformly on $\mathfrak{Z}^{(0)}$.

Then there exist positive constants a and b and a function V on $(\mathfrak{U}^{(0)})_b$ to $\mathfrak{V}$ with the following properties:

(C_1) $(\mathfrak{Z}^{(0)})_a$ is in $\mathfrak{Z}_0$;

(C_2) for every u in $(\mathfrak{U}^{(0)})_b$, $V(u)$ is the unique solution of the equation $G(u\ \ v) = v_*$ for which (u, v) is in $(\mathfrak{Z}^{(0)})_a$;

(C_3) the function V is of class $\mathfrak{C}^{(n)}$ on the region $(\mathfrak{U}^{(0)})_b$.

In case the set $\mathfrak{Z}^{(0)}$ is self-compact, the hypotheses (H_5) to (H_8) may be omitted.

We reduce this theorem to Theorem VI by setting $\mathfrak{P}=\mathfrak{Z}^{(0)}$, $p=(u, v)$, where (u, v) is a point of $\mathfrak{Z}^{(0)}$. Corresponding to this range $\mathfrak{P}$ and the spaces $\mathfrak{U}$ and $\mathfrak{V}$, we define the spaces $\mathfrak{X}$ and $\mathfrak{Y}$ as at the beginning of this section. The initial solution (x_0, y_0) is defined by

$$x_0(p) = u\text{-component of } p \equiv u_p,$$

$$y_0(p) = v\text{-component of } p \equiv v_p.$$

The region $\mathfrak{W}_0$ consists of all points $w=(x, y)$ for which there is a uniform neighborhood $(w(p))_d$ in $\mathfrak{Z}_0$ for every p. Then by Theorem VI there exist positive constants a and b, and a function V_1 on $\mathfrak{P}(u_p)_b$ to $\mathfrak{V}$ such that

(1) $((u_p)_b, (v_p)_a)$ is contained in $\mathfrak{Z}_0$ for every p;

(2) for every p and every u in $(u_p)_b$, $V_1(p, u)$ is the unique solution of the equation $G(u, v)=v_*$ having v in $(v_p)_a$;

(3) the function V_1 is of class $\mathfrak{C}^{(n)}$ on $(u_p)_b$ uniformly on $\mathfrak{P}(u_p)_b$.

We have now to show that when the constants a and b are replaced by constants a' and b' sufficiently small, the function V_1 is independent of p when u is in $(\mathfrak{U}^{(0)})_{b'}$ and yields the unique solution having (u, v) in $(\mathfrak{Z}^{(0)})_{a'}$. The hypothesis (H_7), put in explicit form, reads as follows: for every positive number ϵ there is a positive δ such that, when the distance of two points u_1 and u_2 of $\mathfrak{U}^{(0)}$ is less than δ, we have $\|V^{(0)}(u_1)-V^{(0)}(u_2)\|<\epsilon$. Let δ_1 correspond to $\epsilon=a/2$, and let a' be the smallest of the numbers $a/2$, $\delta_1/2$, and b. Then since the function V_1 is continuous on $(u_p)_b$ uniformly on $\mathfrak{P}(u_p)_b$, the constant a may be replaced by a' in the statements above if b is replaced by a constant b' sufficiently small. Furthermore we may take $b'\leqq a'$. Now suppose (u, v_1) and (u, v_2) are two solutions of the equation $G(u, v)=v_*$, both in $(\mathfrak{Z}^{(0)})_{a'}$. That is, (u, v_1) is in a neighborhood $((u_{01}, v_{01}))_{a'}$ and (u, v_2) is in a neighborhood $((u_{02}, v_{02}))_{a'}$, where (u_{01}, v_{01}) and (u_{02}, v_{02}) are points of $\mathfrak{Z}^{(0)}$. Then we have, in succession,

$$\|u_{01} - u_{02}\| < 2a' \leqq \delta_1,$$

$$\|v_{01} - v_{02}\| < \epsilon = a/2,$$

$$\|v_{01} - v_2\| < a/2 + a' \leqq a,$$

so that v_1 and v_2 are both in $(v_{01})_a$. Therefore $v_1=v_2=V_1(p, u)$, where $p=(u_{01},v_{01})$. Hence on the region $(\mathfrak{U}^{(0)})_{b'}$ we may set $V_1(u, p)=V(u)$.

Finally, from the differentiability properties of the function V_1 we find that the function V is of class $\mathfrak{C}^{(n)}$ on the region $(\mathfrak{U}^{(0)})_{b'}$.

In case the set $\mathfrak{Z}^{(0)}$ is self-compact, the hypotheses (H_5) and (H_6) follow from Lemmas 5.1 and 5.2. The hypothesis (H_7) is obtained immediately by an indirect proof. To obtain (H_8) from Lemma 5.5, we need to show first

that the reciprocal L of d_vG has the same continuity as d_vG on the set $\mathfrak{Z}^{(0)}$. The proof for this is formally the same as that in Lemma 16.2 of Paper I.

V. Extension of the domain of definition of a function, with preservation of a Lipschitz condition

16. In an existence theorem for the solution of a differential equation, it is desirable to show that the solution extends to the boundary of the domain of definition of the functions involved in the differential equation. As Professor Jackson has remarked,* this can easily be done by extending the domain of definition of the functions to include the whole space of the dependent variables. However, this method yields more restricted results than can be obtained by other methods.†

The extension of the domain of definition can readily be carried out in our abstract spaces in case the original domain is a neighborhood. Extensions from more general domains with preservation of continuity have been given,‡ but it does not seem possible to prove that those extensions preserve a Lipschitz condition.

Consider a linear metric space $\mathfrak{X}$ and a complete linear metric space $\mathfrak{Y}$, and a function F on a region $\mathfrak{X}_0$ to $\mathfrak{Y}$. We say that F satisfies a Lipschitz condition with constant k on $\mathfrak{X}_0$ in case for every x_1 and x_2 in $\mathfrak{X}_0$ we have

$$\|F(x_1) - F(x_2)\| \leqq k\|x_1 - x_2\|.$$

Theorem VIII. *If a function F on a neighborhood $(x_0)_a$ to $\mathfrak{Y}$ satisfies a Lipschitz condition with constant k on $(x_0)_a$, then there exists a function G on $\mathfrak{X}$ to $\mathfrak{Y}$, equal to F on $(x_0)_a$, and satisfying a Lipschitz condition with constant $2k$ on $\mathfrak{X}$.*

If the function F is on $(x_0)_a\mathfrak{P}$ to $\mathfrak{Y}$, we evidently have the theorem for each p of $\mathfrak{P}$, whether k depends on p or not. For example, $\mathfrak{P}$ might be a region $\mathfrak{W}_0$ of a metric space $\mathfrak{W}$. It is also evident that the function G of the theorem is not unique, and that the constant $2k$ for G is larger than necessary (at least in most cases).

If a function F on a set $\mathfrak{X}^{(0)}$ to $\mathfrak{Y}$ is continuous on $\mathfrak{X}^{(0)}$ uniformly on $\mathfrak{X}^{(0)}$, then a function H on $\mathfrak{X}^{(0)}$ plus its boundary is uniquely defined by the

* See Annals of Mathematics, (2), vol. 23 (1922), p. 75.

† Cf. Bliss, *The solutions of differential equations of the first order as functions of their initial values*, Annals of Mathematics, (2), vol. 6 (1905), p. 49.

‡ Cf. Tietze, Crelle's Journal, vol. 145 (1915), p. 9; de la Vallée Poussin, *Intégrales de Lebesgue*, Sec. 125.

properties that H equals F on $\mathfrak{X}^{(0)}$, and H is continuous on $\mathfrak{X}^{(0)}$ plus its boundary. This has been shown by Fréchet* by the usual methods for the case when $\mathfrak{Y}$ is the real number system, and the proof is exactly the same for our general case.

Thus we have a uniquely determined function H on $[x_0]_a$ to $\mathfrak{Y}$ (where $[x_0]_a$ denotes the neighborhood $(x_0)_a$ plus its boundary). It is readily verified that the same constant k is valid for H on $[x_0]_a$ as for F on $(x_0)_a$. We now define the function G, for points x not in the set $[x_0]_a$, by the formula

$$G(x) = H\left(x_0 + \frac{(x - x_0)a}{\|x - x_0\|}\right). \tag{16.1}$$

To show that G satisfies a Lipschitz condition, we simplify the formulas by the transformation $x = x_0 + \bar{x}a$. This makes the set $[x_0]_a$ correspond to the set $[x_*]_1$, and the formula (16.1) becomes

$$G(x) = H\left(\frac{x}{\|x\|}\right)$$

for points x not in $[x_*]_1$. Take first the case $\|x_1\| > 1$, $\|x_2\| > 1$. We have

$$\|G(x_1) - G(x_2)\| = \left\| H\left(\frac{x_1}{\|x_1\|}\right) - H\left(\frac{x_2}{\|x_2\|}\right)\right\|$$

$$\leqq k\left\{\left\|\frac{x_1}{\|x_1\|} - \frac{x_2}{\|x_2\|}\right\|\right\} \leqq k\left\{\left\|\frac{x_1}{\|x_1\|} - \frac{x_2}{\|x_1\|}\right\| + \left\|\frac{x_2}{\|x_1\|} - \frac{x_2}{\|x_2\|}\right\|\right\}$$

$$= \frac{k}{\|x_1\|}\left\{\|x_1 - x_2\| + \big|\,\|x_2\| - \|x_1\|\,\big|\right\} < 2k\|x_1 - x_2\|.$$

The manipulation is similar in case one of the points is in $[x_*]_1$, while the other is outside.

VI. Imbedding theorems for differential equations

The equations to be considered are called differential equations for convenience; they include both ordinary differential equations and Volterra integral equations as special cases. The form of the equations is the same as in Paper I, but a different kind of Lipschitz condition is assumed. Consequently a new convergence proof is required.† However, Theorems IV and V

* Bulletin des Sciences Mathématiques, (2), vol. 48 (1924), p. 171.

† For an illuminating discussion of the close connection between the two types of theorems, see Evans, Cambridge Colloquium, pp. 52–54.

of Paper I can be and are applied to the differential equations considered in this section, the new convergence proof being essential in showing the existence of a reciprocal for $d_y G$.

17. **Notations.** We consider a bounded and measurable set $\mathfrak{R}^{(0)}$ of points of the real axis $\mathfrak{R}$, and a linear metric space $\mathfrak{B}$. Let $\mathfrak{Y}$ be a class of functions y on $\mathfrak{R}^{(0)}$ to $\mathfrak{B}$, constituting a complete linear metric space of type A_0, with $\mathfrak{P}=\mathfrak{R}^{(0)}$, and $\sigma(p)=1$ for every p. Let r_0 be a fixed point of $\mathfrak{R}^{(0)}$. For a point y of the space $\mathfrak{Y}$ we shall set

$$\int^r \|y(r)\| = \text{the greatest lower bound of } \left| \int_{r_0}^r R(r)dr \right|,$$

for all functions R on $\mathfrak{R}^{(0)}$ to $\mathfrak{R}$ which are continuous on $\mathfrak{R}^{(0)}$ and such that $R(r) \geqq \|y(r)\|$ for every r. The integral on the right is understood to be taken over the points of $\mathfrak{R}^{(0)}$ contained in the interval $(r_0 r)$. We might suppose the function R to be merely integrable in the sense of Lebesgue, but that would unnecessarily strengthen the hypotheses of the theorems to follow. In many special cases the function $\|y(r)\|$ is itself measurable, but that is not necessarily so in general. In case the set $\mathfrak{R}^{(0)}$ has measure zero, all the results are trivial, but still valid.

18. **General imbedding theorems.** The first theorem corresponds to Theorem I of Paper I.

THEOREM IX. *Let the point y_0 of $\mathfrak{Y}$ and the region $\mathfrak{X}_0$ of the metric space $\mathfrak{X}$, and the function F on $\mathfrak{X}_0(y_0)_a$ to $\mathfrak{Y}$ be such that*

(H_1) *for every x in $\mathfrak{X}_0$ there is a constant $k_x>0$ such that, whenever y_1 and y_2 are in $(y_0)_a$ and r is in $\mathfrak{R}^{(0)}$ we have*

$$\|F(x,y_1 \mid r) - F(x,y_2 \mid r)\| \leqq k_x \int^r \|y_1(r) - y_2(r)\| \,;$$

(H_2) *for every x in $\mathfrak{X}_0$ we have*

$$\|F(x,y_0) - y_0\| < ae^{-k_x d}$$

where d is the diameter of the set $\mathfrak{R}^{(0)}$.

Then there exists a unique function Y on $\mathfrak{X}_0$ to $(y_0)_a$ such that

$$Y(x) = F(x,Y(x))$$

for every x in $\mathfrak{X}_0$.

Following the method of Part V, we could in many special cases extend the range of definition of the function F to include the whole $\mathfrak{Y}$ space. The hypothesis (H_2), which serves to make sure that all the approximations and

their limit are in $(y_0)_a$, could then be omitted. However, the conclusion would also have to be modified.

It is noticeable that the nature of the range $\mathfrak{X}_0$ is not essential in the theorem, and the theorem is in fact unchanged in content if $\mathfrak{X}_0$ is replaced by a general range $\mathfrak{P}$ or else omitted altogether. The case is otherwise in the later theorems where we consider continuity and differentiability with respect to x. In applications, x may represent initial values or parameters or a combination of the two. In the proof of the present theorem we shall omit to write the argument x.

As usual we define a sequence of approximations by the equations

$$y_1 = F(y_0), \quad y_{n+1} = F(y_n).$$

To show that all the approximations are surely defined, we have

$$\|y_1 - y_0\| = c < ae^{-kd} < a, \ \|y_{n+1}(r) - y_n(r)\| \leqq k \int^r \|y_n(r) - y_{n-1}(r)\|,$$

from which, by induction,

$$(18.1) \qquad \|y_{n+1}(r) - y_n(r)\| \leqq \frac{c(k\,|\,r - r_0\,|\,)^n}{n!}\,.$$

Therefore we have, for every n,

$$\|y_n - y_0\| \leqq ce^{kd} < a.$$

The inequality (18.1) shows at the same time the convergence of the sequence $\{y_n\}$ to an element y of $\mathfrak{Y}$. To show that this limit y is a solution of the equation, we have

$$\begin{aligned}\|y(r) - F(y\,|\,r)\| &\leqq \|y(r) - y_{n+1}(r)\| + \|F(y_n\,|\,r) - F(y\,|\,r)\| \\ &\leqq \|y - y_{n+1}\| + k\int^r \|y_n(r) - y(r)\| \leqq \|y - y_{n+1}\| + kd\|y_n - y\|.\end{aligned}$$

To show the uniqueness of the solution we have, if y_1 and y_2 are solutions,

$$\|y_1(r) - y_2(r)\| = \|F(y_1\,|\,r) - F(y_2\,|\,r)\| \leqq k\int^r \|y_1(r) - y_2(r)\|,$$

and by induction,

$$\|y_1(r) - y_2(r)\| \leqq \frac{(k\,|\,r - r_0\,|\,)^n}{n!}\|y_1 - y_2\|.$$

For n sufficiently large, this contradicts $y_1 \neq y_2$.

The next theorem corresponds to Lemma 16.1 of Paper I.

THEOREM X. *Suppose the function G on $\mathfrak{Y}$ to $\mathfrak{Y}$ is distributive on $\mathfrak{Y}$ and is such that there exists a constant M for which*

$$\|G(y \mid r)\| \leqq M \int^r \|y(r)\|$$

for every r and y. Then the function K on $\mathfrak{Y}$ to $\mathfrak{Y}$ defined by

$$K(y) = y - G(y)$$

is linear on $\mathfrak{Y}$ and has a reciprocal L. Moreover the modulus of L is not greater than e^{Md}, where d is the diameter of $\mathfrak{R}^{(0)}$.

The linearity of K is evident. If we apply Theorem IX to the equation

$$y = G(y) + y', \tag{18.2}$$

we find that there exists a unique function L on $\mathfrak{Y}$ to $\mathfrak{Y}$ such that $K(L(y')) = y'$. From the uniqueness we have $L(K(y)) = L(y') = y$.

It remains to show that L is linear. The distributive property of L follows from the form of equation (18.2) and the fact that L is its unique solution. To show that L is modular, we consider the series

$$\sum_{n=0}^{\infty} G_n(y), \text{ where } G_0(y) = y, \quad G_{n+1}(y) = G(G_n(y)),$$

of which L is the sum. By induction we readily find that

$$\|G_n(y \mid r)\| \leqq \frac{(M \mid r - r_0 \mid)^n}{n!} \|y\|$$

and hence $\|L(y)\| \leqq e^{Md} \|y\|$, where d is the diameter of $\mathfrak{R}^{(0)}$.

The next theorem has similarities to Theorems II^{I}, IV^{I}, and V^{I} (of Paper I), and is proved on the basis of Theorems IV^{I}, V^{I}, and Theorem X.

THEOREM XI. *Let $\mathfrak{X}$ be a linear metric space, and let $\mathfrak{W}_0$ be a convex region of the space $\mathfrak{W} = (\mathfrak{X}, \mathfrak{Y})$. Suppose the point $w_0 = (x_0, y_0)$ of $\mathfrak{W}_0$ and the function F on $\mathfrak{W}_0$ to $\mathfrak{Y}$ are such that*

(H_1) *(x_0, y_0) is a solution of the equation*

$$y = F(x, y)\ ; \tag{18.3}$$

(H_2) *F is of class $\mathfrak{C}^{(n)}$ on $\mathfrak{W}_0$;*

(H_3) *the differential d_yF satisfies the condition that for every x there exists a constant M_x such that, whenever (x, y) is in $\mathfrak{W}_0$, we have*

$$\|d_yF(x, y; dy \mid r)\| \leqq M_x \int^r \|dy(r)\|$$

for every r of $\mathfrak{R}^{(0)}$ and every dy in $\mathfrak{Y}$.

Then there exist a unique region $\mathfrak{X}_0$ containing the point x_0 and a unique function Y on $\mathfrak{X}_0$ to $\mathfrak{Y}$ with the following properties:

(C_1) *$\mathfrak{X}_0$ is connected;*

(C_2) *for every x in $\mathfrak{X}_0$, $(x, Y(x))$ is in $\mathfrak{W}_0$, and satisfies the equation* (18.3);

(C_3) *Y is of class $\mathfrak{C}^{(n)}$ on $\mathfrak{X}_0$;*

(C_4) *if x' is a boundary point of the region $\mathfrak{X}_0$, and if $Y(x)$ has a limit y' for $x=x'$, then (x', y') is a boundary point of the region $\mathfrak{W}_0$.*

To apply Theorems IV^{I} and V^{I}, we set

$$y - F(x,y) = G(x,y).$$

Then by Theorem X, d_yG has a reciprocal for every point (x, y) in $\mathfrak{W}_0$. Hence every point of $\mathfrak{W}_0$ is an ordinary point for the function G. Also there is a unique sheet $\mathfrak{W}^{(0)}$ of solutions, with the properties described in Theorem V^{I}.

We next show that there cannot be two solutions y_1 and y_2 for a given x, so that the sheet $\mathfrak{W}^{(0)}$ is single-valued. For, suppose that there were two solutions. Then by Theorem I (§ 4), Taylor's theorem,* and the hypothesis (H_3) we have

$$\begin{aligned}\|y_1(r) - y_2(r)\| &= \|F(x,y_1\,|\,r) - F(x,y_2\,|\,r)\| \\ &= \left\| \int_0^1 d_yF(x, y_2 + (y_1 - y_2)s;\, y_1 - y_2\,|\,r)ds \right\| \\ &\leqq M_x \int^r \|y_1(r) - y_2(r)\|.\end{aligned}$$

From this we obtain by induction

$$\|y_1(r) - y_2(r)\| \leqq \|y_1 - y_2\| \frac{(M_x\,|\,r - r_0\,|\,)^n}{n!},$$

which is a contradiction for n sufficiently large.

By the definition of a sheet, the projection of $\mathfrak{W}^{(0)}$ on the space $\mathfrak{X}$ is a connected region $\mathfrak{X}_0$. Then the sheet $\mathfrak{W}^{(0)}$ defines a single-valued function Y on $\mathfrak{X}_0$ to $\mathfrak{Y}$ satisfying the conclusions (C_1) to (C_3).

A point (x', y') described in (C_4) is certainly a boundary point of the sheet $\mathfrak{W}^{(0)}$, and hence of the region $\mathfrak{W}_0$, since all the points of $\mathfrak{W}_0$ are ordinary.

Suppose there were two regions $\mathfrak{X}_{01}$ and $\mathfrak{X}_{02}$ and corresponding functions Y_1 and Y_2, satisfying the conclusions of the theorem. Then $Y_1 = Y_2$ on the

* See Paper II.

region common to $\mathfrak{X}_{01}$ and $\mathfrak{X}_{02}$. Let x_1 be a point of $\mathfrak{X}_{01}$, connected to x_0 by a function X continuous on the interval (01) to $\mathfrak{X}_{01}$. Let s' be the upper bound of the points s on (01) such that $X(s)$ is in $\mathfrak{X}_{02}$. If $X(s')$ is not in $\mathfrak{X}_{02}$ (which is certainly true if $s' \neq 1$), then $X(s')$ is a boundary point of $\mathfrak{X}_{02}$, and

$$\lim_{x=X(s')} Y_2(x) = Y_1(X(s')).$$

Hence by (C_4), $(X(s'), Y_1(X(s')))$ is a boundary point of the region $\mathfrak{W}_0$, which contradicts (C_2). Hence all the points x_1 of $\mathfrak{X}_{01}$ are contained in $\mathfrak{X}_{02}$. Similarly $\mathfrak{X}_{01}$ contains $\mathfrak{X}_{02}$.

The following modification of the last theorem is interesting, and follows from Theorem X and Theorem IV^{I}.

THEOREM XI′. *In case the hypothesis (H_3) is satisfied only at the initial solution (x_0, y_0), then there exist positive constants a and b and a unique function Y on $(x_0)_b$ to $(y_0)_a$ with the properties (C_2) and (C_3).*

In stating still another theorem following from Theorems X and IV^{I}, we consider a space $\mathfrak{Z}$ of functions z on $\mathfrak{R}^{(0)}$ to a linear metric space $\mathfrak{U}$. The space $\mathfrak{Z}$ is to have all the properties assumed for the space $\mathfrak{Y}$, except that it need not be complete.

THEOREM XII. *Consider a function F on a region $(\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z})_0$ of the composite space $(\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z})$, to $\mathfrak{Y}$, a function H on a region $(\mathfrak{X}, \mathfrak{Y})_1$ of the space $(\mathfrak{X}, \mathfrak{Y})$, to $\mathfrak{Z}$, and a point (x_0, y_0, z_0), with the following properties:*

(H_1) (x_0, y_0) is in $(\mathfrak{X}, \mathfrak{Y})_1$, and (x_0, y_0, z_0) is in $(\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z})_0$;

(H_2) $H(x_0, y_0) = z_0$, $F(x_0, y_0, z_0) = y_*$;

(H_3) F and H are of class $\mathfrak{C}^{(n)}$ on their respective domains of definition [uniformly on those domains];

(H_4) the differential $d_yF(x_0, y_0, z_0; dy)$ has a reciprocal L_0;

(H_5) there exists a positive constant k such that, for every r in $\mathfrak{R}^{(0)}$, every dy in $\mathfrak{Y}$ and every dz in $\mathfrak{Z}$, we have

$$\|d_yH(x_0, y_0; dy \mid r)\| \leqq k \int^r \|dy(r)\|,$$

$$\|d_zF(x_0, y_0, z_0; dz \mid r)\| \leqq k\|dz(r)\|,$$

$$\|L_0(dy \mid r)\| \leqq k\|dy(r)\|.$$

Then there exist positive constants a and b, and a unique function Y on $(x_0)_b$ to $(y_0)_a$, with the following properties:

(C_1) the function

$$G(x, y) = F(x, y, H(x, y))$$

is defined for x in $(x_0)_b$, and y in $(y_0)_a$;

(C_2) *for every x in $(x_0)_b$ the point $(x, Y(x))$ satisfies the equation*

$$G(x,y) = F(x,y,H(x,y)) = y_* ;$$

(C_3) *the function Y is of class $\mathfrak{C}^{(n)}$ on $(x_0)_b$ [uniformly on $(x_0)_b$].*

Since a region consists wholly of interior points, and the function H is continuous, the conclusion (C_1) is evidently true whenever the constants a and b are sufficiently restricted. The function G is of class $\mathfrak{C}^{(n)}$ on $(x_0)_b(y_0)_a$ [uniformly on $(x_0)_b(y_0)_a$] by Lemma 15.3 of Paper I. To show that d_yG has a reciprocal at (x_0, y_0), set

$$K_1(dy) = d_yF(x_0,y_0,z_0;dy),$$

$$K_2(dy) = d_zF(x_0,y_0,z_0;\, d_yH(x_0,y_0;dy)),$$

$$K_3(dy) = d_yG(x_0,y_0;dy),$$

$$K_4(dy) = -\,L_0(K_2(dy)).$$

Then $K_3=K_1+K_2$, and L_0 is the reciprocal of K_1. The linear function K_4 satisfies the conditions of Theorem X, since

$$\|K_4(dy\mid r)\| \leqq k^3 \int^r \|dy(r)\|.$$

Let K_5 be the reciprocal of $dy-K_4(dy)$. Then $K_5(L_0(dy))$ is the reciprocal of $K_3(dy)=d_yG(x_0,\ y_0;\ dy)$. For we have (omitting to write parentheses)

$$K_5L_0K_3dy = K_5L_0K_1dy + K_5L_0K_2dy = K_5dy - K_5K_4dy = dy,$$

$$K_3K_5L_0dy = K_1K_5L_0dy + K_2K_5L_0dy = K_1(K_5 - K_4K_5)L_0dy = K_1L_0dy = dy.$$

If now the constants a and b are sufficiently restricted, the conclusions (C_2) and (C_3) follow at once from Theorem IV^{I}.

In case the hypothesis (H_5) of the last theorem holds for every (x, y, z) for which d_yF has a reciprocal, we can derive from Theorem V^{I} a theorem on the unique maximal sheet of solutions through a given solution. The statement and proof of this are fairly obvious. Here the sheet need not be single-valued as it is in Theorem XI.

19. **A special imbedding theorem.** In this subsection we shall consider a case in which the spaces $\mathfrak{U}$ and $\mathfrak{B}$ are restricted to be euclidean spaces of m and k dimensions, respectively. The space $\mathfrak{X}$ need not be restricted. For the space $\mathfrak{Y}$ (which is to be of type A_0, as before, with $\sigma(p)\equiv 1$) we take the class of all functions y on $\mathfrak{R}^{(0)}$ to $\mathfrak{B}$ which are bounded and measurable on $\mathfrak{R}^{(0)}$, i.e., all those functions y whose components are bounded and measurable on $\mathfrak{R}^{(0)}$. The space $\mathfrak{Z}$ corresponds to the space $\mathfrak{U}$ in exactly the same way.

If distance is properly defined in the spaces $\mathfrak{U}$ and $\mathfrak{V}$, then $\mathfrak{Y}$ and $\mathfrak{Z}$ may be regarded as spaces of type D. For these spaces the integrals

$$\int^{r} y(r)dr, \qquad \int^{r} z(r)dr, \qquad \int^{r} \|y(r)\|dr, \text{ etc.}$$

all exist in the Lebesgue sense, each integral being understood to be taken over the set of points in $\mathfrak{R}^{(0)}$ contained between the points r_0 and r of $\mathfrak{R}^{(0)}$. In case the space $\mathfrak{R}$ enters twice as a component in the range of definition of a function, we shall use the notations $\mathfrak{R}'$ and r' to denote the second component and second argument, respectively.

The form of the equations considered below was suggested by the paper of Hahn already mentioned.* However, our theorem is much more general than Hahn's.

THEOREM XIII. *Consider a point* (x_0, y_0, z_0), *a function* F *on* $\mathfrak{R}^{(0)}(x_0, y_0(r), z_0(r))_c$ *to* $\mathfrak{V}$, *and a function* H *on* $\mathfrak{R}^{(0)}\ \mathfrak{R}'^{(0)}\ (x_0, y_0(r'))_c$ *to* $\mathfrak{U}$, *with the following properties*:

(H_1) *F and H are bounded on their respective domains of definition*;

(H_2) *for every point* (x, v, u) *the function* $F(r, x, v, u)$ *is measurable in* r *on every measurable set on which it is defined, and for every point* (x, v) *the function* $H(r, r', x, v)$ *is measurable in* r *and* r' *together and in* r' *alone, on every measurable set on which it is defined*;

(H_3) *F is of class* $\mathfrak{C}^{(n)}$ *on* $(x_0, y_0(r), z_0(r))_c$ *uniformly on* $\mathfrak{R}^{(0)}\ (x_0, y_0(r), z_0(r))_c$, *and H is of class* $\mathfrak{C}^{(n)}$ *on* $(x_0, y_0(r'))_c$ *uniformly on* $\mathfrak{R}^{(0)}\ \mathfrak{R}'^{(0)}\ (x_0, y_0(r'))_c$;

(H_4) *for every r in* $\mathfrak{R}^{(0)}$ *we have*

$$\int^{r} H(r, r', x_0, y_0(r'))dr' = z_0(r),$$

$$F(r, x_0, y_0(r), z_0(r)) = v_* ;$$

(H_5) *the functional determinant*

$$| F_v(r, x_0, y_0(r), z_0(r)) |$$

(*whose elements are the partial derivatives of the components of F with respect to the components of v*) *is bounded away from zero on the set* $\mathfrak{R}^{(0)}$.

Then there exist positive constants a and b and a unique function Y on $(x_0)_b$ *to* $(y_0)_a$ *such that*

(C_1) *for every x in* $(x_0)_b$ *and r in* $\mathfrak{R}^{(0)}$ *we have*

* Monatshefte für Mathematik und Physik, vol. 14 (1903), p. 326.

$$F\left(r, x, Y(x \mid r), \int^{r} H(r, r', x, Y(x \mid r'))dr'\right) = v_* ;$$

(C_2) *the function* Y *is of class* $\mathfrak{C}^{(n)}$ *on* $(x_0)_b$ *uniformly on* $(x_0)_b$.

We wish to show first of all that for every point (x, y, z) in $(x_0, y_0, z_0)_c$, the function $F(r, x, y(r), z(r))$ is bounded and measurable on $\mathfrak{R}^{(0)}$. This follows from (H_1), (H_2) and (H_3), and a theorem on the measurability of functions of measurable functions.* Hence the function F_1 defined by

$$F_1(x, y, z \mid r) = F(r, x, y(r), z(r))$$

is on $(x_0, y_0, z_0)_c$ to $\mathfrak{Y}$. Similarly $H(r, r', x, y(r'))$ is bounded and measurable on $\mathfrak{R}^{(0)}\ \mathfrak{R}'^{(0)}$ and also is measurable on $\mathfrak{R}'^{(0)}$ for every r in $\mathfrak{R}^{(0)}$. Then by Theorem III of Paper III the function Z defined by

$$Z(x, y \mid r) = \int^{r} H(r, r', x, y(r'))dr' \tag{19.1}$$

is on $(x_0, y_0)_c$ to $\mathfrak{Z}$.

To show that the function F_1 is of class $\mathfrak{C}^{(n)}$ on $(x_0, y_0, z_0)_c$ uniformly on $(x_0, y_0, z_0)_c$, we temporarily replace the spaces $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$ by the more inclusive spaces of type B, consisting of all functions on $\mathfrak{R}^{(0)}$ to $\mathfrak{X}$, $\mathfrak{V}$, and $\mathfrak{U}$, respectively, which are bounded on $\mathfrak{R}^{(0)}$. Then we can apply Lemma 13.2 and Lemma 9.3.

To show that the function Z is of class $\mathfrak{C}^{(n)}$, we consider first the function Z_1 on $(x_0, y_0)_c\mathfrak{R}^{(0)}$ to $\mathfrak{U}$ defined by

$$Z_1(x, y, r) = \int^{r} H(r, r', x, y(r'))dr' = Z(x, y \mid r).$$

If we set $\mathfrak{T} = (\mathfrak{X}, \mathfrak{Y})$, $\mathfrak{W} = (\mathfrak{X}, \mathfrak{V})$, then it is readily verified that the function d_tZ_1 defined by

$$d_tZ_1(x, y, r; dx, dy) = \int^{r} d_wH(r, r', x, y(r'); dx, dy(r'))dr'$$

satisfies the three conditions for the function Z_1 to be of class $\mathfrak{C}'$ on $(x_0, y_0)_c$ uniformly on $(x_0, y_0)_c\mathfrak{R}^{(0)}$, with d_tZ_1 for its differential. The integrand on the right is measurable, since by Lemma 11.1 of Paper I each component is the limit of a convergent sequence of measurable functions.

The proof is completed by induction. Suppose that when H is of class $\mathfrak{C}^{(k)}$, Z_1 is of class $\mathfrak{C}^{(k)}$, and suppose that H is of class $\mathfrak{C}^{(k+1)}$. Then d_wH is

* See Paper III, Theorem II. The theorems of this note are simple extensions and consequences of a theorem in Carathéodory's *Vorlesungen über reelle Funktionen*, pp. 665–66.

of class $\mathfrak{C}^{(k)}$ on $(x_0, y_0(r'))_c$ uniformly $(\mathfrak{R}^{(0)}\mathfrak{R}'^{(0)}\ (x_0, y_0(r'))_c\ \mathfrak{W};\ \|dw\|)$, and linear on $\mathfrak{W}$ uniformly on $\mathfrak{R}^{(0)}\ \mathfrak{R}'^{(0)}\ (x_0, y_0(r'))_c$. Hence by Lemma 7.2, d_wH is of class $\mathfrak{C}^{(k)}$ on $(x_0, y_0(r'))_c\ (x_*, y_*(r'))_c$ uniformly on $\mathfrak{R}^{(0)}\ \mathfrak{R}'^{(0)}$ $(x_0, y_0(r'))_c\ (x_*, y_*(r'))_c$. This is a case exactly like the original, so that d_tZ_1 is of class $\mathfrak{C}^{(k)}$ on $(x_0, y_0, x_*, y_*)_c$ uniformly on $(x_0, y_0, x_*, y_*)_c\ \mathfrak{R}^{(0)}$. Then by another application of Lemma 7.2 we find that d_tZ_1 is of class $\mathfrak{C}^{(k)}$ on $(x_0, y_0)_c$ uniformly $((x_0, y_0)_c\ \mathfrak{R}^{(0)}\ \mathfrak{T};\ \|dt\|)$, from which, by definition, Z_1 is of class $\mathfrak{C}^{(k+1)}$. This completes the induction.

Finally, by application of Theorem I of § 4, we have the result that the function Z on $(x_0, y_0)_c$ to $\mathfrak{Z}$ is of class $\mathfrak{C}^{(n)}$ on $(x_0, y_0)_c$ uniformly on $(x_0, y_0)_c$.

We have next to show that the differential $d_yF_1(x_0, y_0, z_0;\ dy)$ has a reciprocal. From the hypothesis (H_3), we can show that the elements of the matrix

$$K_1(r) \equiv F_v(r, x_0, y_0(r), z_0(r))$$

are bounded on $\mathfrak{R}^{(0)}$, and from (H_2) that they are measurable on $\mathfrak{R}^{(0)}$. Then the hypothesis (H_5) shows that the matrix $K_1(r)$ has a reciprocal matrix $L_1(r)$, each of whose elements is bounded and measurable on $\mathfrak{R}^{(0)}$ Then the reciprocal L_0 of $d_yF_1(x_0, y_0, z_0;\ dy)$ is obviously given by

$$L_0(dy \mid r) = L_1(r)dy(r),$$

where the multiplication on the right hand side is matrix multiplication.

Finally, it is readily verified that there exists a constant k such that, for every r in $\mathfrak{R}^{(0)}$, dy in $\mathfrak{Y}$, and dz in $\mathfrak{Z}$, we have

$$\|L_0(dy \mid r)\| \leqq k\|dy(r)\|,$$

$$\|d_yZ(x_0, y_0;\ dy \mid r)\| = \left\| \int^r d_v H(r, r', x_0, y_0(r'); dy(r'))dr' \right\|$$

$$\leqq k \int^r \|dy(r')\| dr',$$

$$\|d_zF_1(x_0, y_0, z_0;\ dz \mid r)\| = \|d_uF(r, x_0, y_0(r), z_0(r);\ dz(r))\| \leqq k\|dz(r)\|.$$

Thus all the hypotheses of Theorem XII are fulfilled, with F replaced by F_1, and H by Z, so that the theorem follows from Theorem XII.

In cases where the domains of definition of the functions F and H are sufficiently regular, and F_v is always continuous on $\mathfrak{R}^{(0)}=$ a closed interval, it is easy to obtain from Theorem V of Paper I a theorem on the maximal sheet of solutions. We can then amplify the following Theorem XIV, by making use of Theorem V of § 10.

We next consider the case where the solution $Y(x)$ is continuous as a function of r, at the points of a set $\mathfrak{R}^{(1)}$ included in $\mathfrak{R}^{(0)}$. We have the following theorem:

THEOREM XIV. *Suppose that the initial solution y_0, the functions F and H, and the set $\mathfrak{R}^{(1)}$ contained in $\mathfrak{R}^{(0)}$ are such that*
(H_1) *y_0 is continuous in r at the points of $\mathfrak{R}^{(1)}$;*
(H_2) *for every (x, v, u), F is continuous in r at the points of $\mathfrak{R}^{(1)}$ for which it is defined;*
(H_3) *H is continuous in r at the points of $\mathfrak{R}^{(1)}$ uniformly on $\mathfrak{R}'^{(0)}$ $(x_0, y_0(r'))_c$.*
Then the function V on $(x_0)_b\mathfrak{R}^{(0)}$ to $\mathfrak{V}$ defined by

$$V(x,r) = Y(x \mid r)$$

is measurable on $\mathfrak{R}^{(0)}$ and continuous in r at the points of $\mathfrak{R}^{(1)}$ for every x in $(x_0)_b$, and is of class $\mathfrak{C}^{(n)}$ on $(x_0)_b$ uniformly on $(x_0)_b\mathfrak{R}^{(0)}$.

To prove this theorem we apply Theorem IV of § 10, with $\mathfrak{X}' = \mathfrak{X}$, and $\mathfrak{Y}'$ = the subspace of $\mathfrak{Y}$ composed of all those functions y which are continuous at the points of $\mathfrak{R}^{(1)}$. The subspace $\mathfrak{Y}'$ is therefore a space of type D_1.

We have first to show that if the function Z is defined by equation (19.1), and if the function G on $(x_0, y_0)_c$ to $\mathfrak{Y}$ is defined by

$$G(x,y \mid r) = F(r,x,y(r),Z(x,y \mid r)), \tag{19.2}$$

then G as on $(x_0)_b$ $(y_0)_a'$ is to $\mathfrak{Y}'$. Under the hypotheses of the present theorem, the function Z as on $(x_0, y_0)_c$ is to $\mathfrak{Y}'$. From (H_3) of Theorem XIII, and Lemma 12.2 of Paper I, we obtain the fact that F is continuous on $(y_0(r),\ z_0(r))_c$ uniformly on $\mathfrak{R}^{(0)}(x_0,\ y_0(r),\ z_0(r))_c$. From this we readily obtain the desired property of the function G.

Secondly, we have to show that the reciprocal L of $d_yG(x_0,\ y_0;\ dy)$ as on $\mathfrak{Y}'$ is to $\mathfrak{Y}'$. We follow the steps in the proof of the existence of this reciprocal, as given in the proof of Theorem XII. The linear function K_2 on $\mathfrak{Y}$ to $\mathfrak{Y}$ defined by

$$K_2(dy \mid r) = d_uF(r,x_0,y_0(r),z_0(r);\, d_yZ(x_0,y_0;\, dy \mid r))$$

is to $\mathfrak{Y}'$, by Lemma 9.3. For the same reason the elements of the matrix $F_v(r,\ x_0,\ y_0(r),\ z_0(r))$ are continuous in r at the points of $\mathfrak{R}^{(1)}$. The same will consequently be true of the reciprocal matrix $L_1(r)$. Hence if the linear functions L_0 and K_4 are defined by

$$L_0(dy \mid r) = L_1(r)dy(r), \quad K_4(dy) = -L_0(K_2(dy)),$$

then L_0 and K_4 as on $\mathfrak{Y}'$ are to $\mathfrak{Y}'$. The reciprocal K_5 of $dy-K_4(dy)$ is given by the series

$$K_5(dy) = \sum_{n=0}^{\infty} K_{4n}(dy), \quad K_{40}(dy) = dy, \quad K_{4,n+1}(dy) = K_4(K_{4n}(dy)).$$

Hence by Lemma 9.1, K_5 as on $\mathfrak{Y}'$ is to $\mathfrak{Y}'$. Finally the function L, reciprocal to $d_yG(x_0, y_0; dy)$, defined by

$$L(dy) = K_5(L_0(dy)),$$

evidently has the same property.

The last clause of the theorem follows immediately from Theorem I of §4.

The imbedding theorem* for differential equations of the form

$$F(r, z', z) = 0$$

becomes a special case of Theorems XIII and XIV if we set

$$z' = y, \quad z(r) = x + \int_{r_0}^{r} y(r')dr' = x + \int_{r_0}^{r} H(y(r'))dr'.$$

VII. Existence theorems for differential equations

Existence theorems for differential equations in general analysis have been given by E. H. Moore and by T. H. Hildebrandt.† The theorems given here are somewhat different in form. One object of this section is to show how these existence theorems may be derived from the preceding imbedding and implicit function theorems.‡

20. **Notations.** Instead of considering an arbitrary measurable set $\mathfrak{R}^{(0)}$, we now restrict attention to an open *interval* $\mathfrak{R}_0$ of the real axis. For the space $\mathfrak{Y}$ we take the space (of type C_0) consisting of all functions y on $\mathfrak{R}_0$ to $\mathfrak{B}$ which are continuous on $\mathfrak{R}_0$. We assume that the space $\mathfrak{B}$ is complete, so that the space $\mathfrak{Y}$ is also complete.

If we restrict attention to the points of a sub-interval $\mathfrak{R}_0^1$ of $\mathfrak{R}_0$, we obtain from the space $\mathfrak{Y}$ a space $\mathfrak{Y}^1$ consisting of functions y^1 on $\mathfrak{R}_0^1$ to $\mathfrak{B}$. The space $\mathfrak{Y}^1$ is also a complete linear metric space of type C_0. Corresponding to a region $\mathfrak{Y}_0$ of $\mathfrak{Y}$ there is a definitely determined region $\mathfrak{Y}_0^1$ of $\mathfrak{Y}^1$.

* Cf. Bolza, *Vorlesungen über Variationsrechnung*, pp. 179, 185.
Bliss, Bulletin of the American Mathematical Society, vol. 25 (1918), p. 15.

† Moore, Atti di IV Congresso (Rome, 1908), vol. II, p. 98.
Hildebrandt, these Transactions, vol. 18 (1917), p. 73.

‡ Acknowledgements are due to Mr. H. B. Curry of Harvard University for suggestions on this subject.

We let r_0 denote a point of the interval $\mathfrak{R}_0$, and we retain also the other notations of Part VI.

21. **An existence theorem "im kleinen."** We prove the following theorem:

THEOREM XV. *Suppose the region $\mathfrak{Y}_0$ of $\mathfrak{Y}$, the point y_0 of $\mathfrak{Y}_0$, and the function F on $\mathfrak{Y}_0$ to $\mathfrak{Y}$ are such that*

(H_1) *there is a positive constant k such that, for every pair y_1, y_2 of points of $\mathfrak{Y}_0$ and every point r of $\mathfrak{R}_0$ we have*

$$\|F(y_1 \mid r) - F(y_2 \mid r)\| \leqq k \int^r \|y_1(r) - y_2(r)\| \; ;$$

$$(H_2) \qquad y_0(r_0) = F(y_0 \mid r_0).$$

Then there exist an interval $\mathfrak{R}_0^1$ containing the point r_0, and a unique point y^1 in the corresponding region $\mathfrak{Y}_0^1$, such that, for every r of $\mathfrak{R}_0^1$ we have

$$y^1(r) = F(y^1 \mid r).$$

Moreover, $y^1(r_0) = y_0(r_0)$.

In order that the conclusion may have a meaning, the function $F(y|r)$, when r is in $\mathfrak{R}_0^1$, must depend only on the part y^1 of y. That this is so follows from (H_1), which in fact allows us to draw the conclusion that $F(y|r)$ depends only on the values of the function y on the interval (r_0r).

From the definition of a region, there is a neighborhood $(y_0)_a$ contained in $\mathfrak{Y}_0$. Choose positive numbers $b<1$, and $c<(1-b)a$. Then the interval $\mathfrak{R}_0^1$ is chosen so that, for every r in $\mathfrak{R}_0^1$, we have

$$(21.1) \qquad k \mid r - r_0 \mid \leqq b,$$

$$(21.2) \qquad \|F(y_0 \mid r) - F(y_0 \mid r_0)\| \leqq \frac{c}{2}, \qquad \|y_0(r_0) - y_0(r)\| \leqq \frac{c}{2} \cdot$$

Then the hypotheses of Theorem I of Paper I are fulfilled for the function F on the neighborhood $((y_0)_a)^1$.

To deduce the theorem from Theorem IX, we may omit the condition (21.1) on the interval $\mathfrak{R}_0^1$. The conditions (21.2) are retained, with the constant c restricted only the inequality $0<c<ae^{-kd}$, where d is the length of the interval $\mathfrak{R}_0$.

22. **An existence theorem "im grossen."** Consider a point y_0 of the space $\mathfrak{Y}$, and a function F on $\mathfrak{R}_0\mathfrak{V}_r$ to $\mathfrak{V}$, where $\mathfrak{V}_r = (y_0(r))_a$. Suppose that for every

v, F is continuous on its domain of definition, and that for every r and every v_1 and v_2 in $\mathfrak{V}_r$ we have

$$\|F(r,v_1) - F(r,v_2)\| \leqq k\|v_1 - v_2\|.$$

Then we wish to show the existence of a continuous function y on $\mathfrak{R}_0$ (or a part of $\mathfrak{R}_0$) to $\mathfrak{V}$, satisfying the equation

$$(22.1) \qquad y(r) = \int_{r_0}^{r} F(r, y(r))dr + v,$$

where v is a point in $(y_0(r_0))_a$. The integral here is a "Riemann" integral, and certainly exists if the integrand is continuous.* More particularly, we wish to show that a continuous solution y exists on an interval $\mathfrak{R}_0^1$ included in $\mathfrak{R}_0$, such that, if r_1 is an end point of $\mathfrak{R}_0^1$ but not an end point of $\mathfrak{R}_0$, we have

$$\lim_{r=r_1} \|y(r) - y_0(r)\| = a.$$

To do this we can apply Theorem VIII of § 16, i.e., define a function G on $\mathfrak{R}_0\mathfrak{V}$ to $\mathfrak{V}$, equal to F on $\mathfrak{R}_0\mathfrak{V}_r$, continuous on $\mathfrak{R}_0$ for every v, and satisfying the condition

$$\|G(r,v_1) - G(r,v_2)\| \leqq 2k\|v_1 - v_2\|,$$

for every r, v_1, v_2. Theorem VIII does not yield the continuity of the function G on $\mathfrak{R}_0$. That G is continuous on $\mathfrak{R}_0$ can be shown as follows. Let $\{b_n\}$ be a properly monotone increasing sequence of positive numbers approaching the number a. Let G_n be the function formed (in the same way as G) from the function F regarded as defined on $\mathfrak{R}_0(y_0(r))_{b_n}$. Then each G_n is continuous on $\mathfrak{R}_0$ for every v. Furthermore, we have $\|G(r, v) - G_n(r, v)\| \leqq k(a - b_n)$, so that G_n approaches G uniformly on $\mathfrak{R}_0\mathfrak{V}$.

If now we can show that the equation

$$(22.2) \qquad y(r) = \int_{r_0}^{r} G(r, y(r))dr + v$$

has a unique continuous solution, then the part of it falling in the domain of definition of the function F is the solution of equation (22.1) which we desire. The integrand $G(r, y(r))$ above is evidently continuous on $\mathfrak{R}_0$ whenever y is continuous on $\mathfrak{R}_0$, so that the integral surely exists for every y in the space $\mathfrak{Y}$. The existence of a solution of (22.2) is shown in the following theorem.

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* See Paper II, Theorem I.

THEOREM XVI. *Suppose the function F on $\mathfrak{Y}$ to $\mathfrak{Y}$ has the property that there exists a positive constant k such that, for every y_1 and y_2 in $\mathfrak{Y}$ and every r in $\mathfrak{R}_0$ we have*

$$\|F(y_1 \mid r) - F(y_2 \mid r)\| \leqq k \int^r \|y_1(r) - y_2(r)\|.$$

Then there is a unique function y in $\mathfrak{Y}$ such that

$$y = F(y).$$

This theorem follows immediately from Theorem IX. The initial function y_0 may be chosen arbitrarily. Then take $a > e^{kd}\|F(y_0) - y_0\|$, and the solution lies in $(y_0)_a$.

23. **A special case.*** We wish now to derive an existence theorem (im kleinen) for solutions of equations of the form discussed in § 19. Let $\mathfrak{U}$ and $\mathfrak{V}$ be ordinary spaces of m and k dimensions, respectively. As before $\mathfrak{R}_0$ and $\mathfrak{R}_0'$ denote the same linear interval, and r_0 denotes a point in it.

THEOREM XVII. *Consider a function F on $\mathfrak{R}_0\mathfrak{V}_0\mathfrak{U}_0$ to $\mathfrak{V}$, a function H on $\mathfrak{R}_0\mathfrak{R}_0'\mathfrak{V}_0$ to $\mathfrak{U}$, and a point v_0 of $\mathfrak{V}_0$, with the following properties:*

(H_1) *$\mathfrak{U}_0$ contains the origin u_*;*

(H_2) *$F(r_0, v_0, u_*) = v_*$;*

(H_3) *F is of class $\mathfrak{C}'$ on $\mathfrak{V}_0\mathfrak{U}_0$ uniformly on $\mathfrak{R}_0\mathfrak{V}_0\mathfrak{U}_0$, and continuous on $\mathfrak{R}_0$;*

(H_4) *H is bounded on $\mathfrak{R}_0\mathfrak{R}_0'\mathfrak{V}_0$, of class $\mathfrak{C}'$ on $\mathfrak{V}_0$ uniformly on $\mathfrak{R}_0\mathfrak{R}_0'\mathfrak{V}_0$, continuous on $\mathfrak{R}_0$ uniformly on $\mathfrak{R}_0'\mathfrak{V}_0$, and measurable on $\mathfrak{R}_0'$;*

(H_5) *the functional determinant*

$$|F_v(r_0, v_0, u_*)|$$

is different from zero.

Then there exist an interval $\mathfrak{R}_0^1$ containing r_0, a positive constant a, and a unique continuous function y^1 on $\mathfrak{R}_0^1$ to $(v_0)_a$ such that, for every r in $\mathfrak{R}_0^1$,

$$F(r, y^1(r), \int_{r_0}^{r} H(r, r', y^1(r'))dr') = v_*.$$

Moreover, $y^1(r_0) = v_0$.

The first step in the proof is to apply the implicit function theorems to the equation $F(r, v, u) = v_*$ to show that there exist positive constants a and b, and a unique solution $V(r, u)$ on $(r_0, u_*)_b$ to $(v_0)_a$, and that the function V so obtained is continuous on $(r_0)_b$, and is of class $\mathfrak{C}'$ on $(u_*)_b$ uniformly

* Cf. Hahn, loc. cit.

on $(r_0, u_*)_b$. The simplest procedure seems to be to apply Theorem III of Paper I to show the existence and uniqueness of the solution V. The continuity of V on $(r_0, u_*)_b$ is then obtained by Theorem II of Paper I. From (H_3) and the statement just made we can show that the functional determinant

$$(23.1) \qquad | F_v(r, V(r,u), u) |$$

is continuous on the neighborhood $(r_0, u_*)_b$. Hence if b is sufficiently small, the functional determinant (23.1) is bounded away from zero on that neighborhood, by (H_5). Therefore we can apply Theorem VI of § 14 to show that (when b is sufficiently small) V is of class $\mathfrak{C}'$ on $(u_*)_b$ uniformly on $(r_0,u_*)_b$.

We next consider the function H. First, $H(r, r', y^1(r'))$ is measurable in r' for every continuous function y^1 on $(r_0)_b$ to $\mathfrak{V}$ which is interior to $\mathfrak{V}_0$. This follows from Theorem II of Paper III. Secondly, the function

$$U(r, y^1) = \int_{r_0}^{r} H(r, r', y^1(r'))dr'$$

is continuous on $(r_0)_b$.

For the initial function y_0^1 on $(r_0)_b$ to $\mathfrak{V}$ we take $y_0^1(r) = v_0$ for every r. Let

$$G(y^1 \mid r) = V(r, U(r, y^1)).$$

Then G is on $(y_0^1)_a$ to $\mathfrak{Y}^1$. By two applications of Taylor's theorem we find

$$\|G(y_1^1 \mid r) - G(y_2^1 \mid r)\| \leqq M^2 \int_{r_0}^{r} \|y_1^1(r') - y_2^1(r')\| dr',$$

where M is greater than the moduli of $d_u V$ and $d_v H$. Hence we can apply Theorem XV to secure the desired result, with possibly a further restriction on the interval $(r_0)_b = \mathfrak{R}_0^1$, and consequent alteration of Y^1.

VIII. Special properties of linear equations

24. In this section we seek to generalize the fundamental property of solutions of systems of homogeneous linear differential equations of the first order which states that a set of solutions linearly independent at a given point are linearly independent at every point of their interval of existence.

As a basis for deriving this property, we state an elegant theorem relating to a certain class of linear transformations defined by a linear equation. This theorem is due to Professor T. H. Hildebrandt, although the proof for the more general case stated below is my own.

Let $\mathfrak{R}^{(0)}$ be a bounded and measurable linear set of the real axis $\mathfrak{R}$. Let $\mathfrak{B}$ be the complete linear metric space of type B consisting of all functions v on a range $\mathfrak{P}$ to $\mathfrak{R}$ which are bounded on $\mathfrak{P}$. Let $\mathfrak{Y}$ be the complete linear metric space of type A_0 consisting of all functions y on $\mathfrak{R}^{(0)}$ to $\mathfrak{B}$ which are bounded on $\mathfrak{R}^{(0)}$, and are such that $y(r|p)$ is measurable on $\mathfrak{R}^{(0)}$ for every p.

THEOREM XVIII. *Let K be a function on $\mathfrak{B}\mathfrak{R}^{(0)}$ to $\mathfrak{B}$ which is linear on $\mathfrak{B}$ uniformly on $\mathfrak{R}^{(0)}$, and such that the function K_1 on $\mathfrak{Y}$ defined by*

$$K_1(y \mid r) = K(y(r),r)$$

is to $\mathfrak{Y}$. Then there exists a unique function V on $\mathfrak{B}\mathfrak{R}^{(0)}\mathfrak{R}^{(0)}$ to $\mathfrak{B}$ such that

(C_1) *for each v and r_1, $V(v, r_1, r)$, as a function of r on $\mathfrak{R}^{(0)}$, belongs to the class $\mathfrak{Y}$;*

(C_2) *for every v in $\mathfrak{B}$, and every r_1 and r in $\mathfrak{R}^{(0)}$, we have*

$$V(v,r_1,r) = \int_{r_1}^{r} K(V(v,r_1,r'),r')dr' + v,$$

where the integral is taken over the points of $\mathfrak{R}^{(0)}$ contained in the interval (r_1r).

This function V has the further properties

(C_3) *V is linear on $\mathfrak{B}$ uniformly on $\mathfrak{R}^{(0)}$ $\mathfrak{R}^{(0)}$;*

(C_4) *for every v in $\mathfrak{B}$, and every r_1, r_2, r_3 in $\mathfrak{R}^{(0)}$ we have*

$$V(v,r_1,r_3) = V(V(v,r_1,r_2),r_2,r_3) \; ; \tag{24.1}$$

(C_5) *for every r_1, r_2 in $\mathfrak{R}^{(0)}$ the linear function $V(v, r_1, r_2)$ has a reciprocal, viz., $V(v, r_2, r_1)$.*

The conclusions (C_1), (C_2), and (C_3) follow at once from Theorem X of § 18. To obtain (C_4) we have

$$\begin{aligned} V(v,r_1,r) &= \int_{r_2}^{r} K(V(v,r_1,r'),r')dr' \\ &\quad + \int_{r_1}^{r_2} K(V(v,r_1,r'),r')dr' + v \\ &= \int_{r_2}^{r} K(V(v,r_1,r'),r')dr' + V(v,r_1,r_2). \end{aligned}$$

Then since the equation

$$y(r) = \int_{r_2}^{r} K(y(r'),r')dr' + V(v,r_1,r_2)$$

has a unique solution, viz., $y(r) = V(V(v, r_1, r_2), r_2, r)$, we have the required result. (C_5) is an obvious corollary of (C_4).

The proof just given for the property (C_4) is valid on the interval of existence of the solution, whether the function K is linear or not. The existence and uniqueness may be obtained from Theorem XVI of § 22. When K is linear, another proof of the property (C_4) may be given by using formulas analogous to (24.1) for the terms of the infinite series obtained by the method of successive approximations, of which the function V is the sum.

THEOREM XIX. *Let K_1, K_2, and L_1 be linear functions on $\mathfrak{V}\mathfrak{R}^{(0)}$ to $\mathfrak{V}$ with the same properties as the function K of Theorem* XVIII, *and let L_1 be reciprocal to K_1, for every r of $\mathfrak{R}^{(0)}$. Then there exists a unique function Y on $\mathfrak{V}\mathfrak{R}^{(0)}$ to $\mathfrak{Y}$ such that*

(C_1) *for every v, r_1, and r, we have*

$$K_1(Y(v,r_1 \mid r),r) = \int_{r_1}^{r} K_2(Y(v,r_1 \mid r'),r')dr' + v.$$

The function Y has the further properties

(C_2) *Y is linear on $\mathfrak{V}$ uniformly on $\mathfrak{R}^{(0)}$;*

(C_3) *if $Y(v, r_1|r_2)=v_*$ for a certain r_1, r_2, then $v=v_*$;*

(C_4) *if the points $v_1, v_2, \cdots, v_n$ are linearly independent, so are the points*

$$Y(v_1,r_1 \mid r_2), Y(v_2,r_1 \mid r_2), \cdots, Y(v_n,r_1 \mid r_2),$$

*for every r_1 and r_2.**

Under the transformation of $\mathfrak{Y}$ into $\mathfrak{Y}$ defined by

$$y'(r) = K_1(y(r),r), \quad y(r) = L_1(y'(r),r),$$

the equations

$$K_1(y(r),r) = \int_{r_1}^{r} K_2(y(r'),r')dr' + v,$$

$$y'(r) = \int_{r_1}^{r} K_2(L_1(y'(r'),r'),r')dr' + v$$

are equivalent. Then if we set

$$K(v,r) = K_2(L_1(v,r),r), \quad Y(v,r_1 \mid r) = L_1(V(v,r_1,r),r),$$

we have our theorem from Theorem XVIII. We obtain the property (C_3) from the fact that both L_1 and V have reciprocals, and (C_4) follows from (C_3) and the linearity of Y.

* Cf. Hahn, loc. cit., pp. 330–332.

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CONCERNING ACYCLIC CONTINUOUS CURVES*

BY

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In this paper, we propose to use the word *acyclic* in place of the phrase *containing no simple closed curve*. That is, an acyclic continuous curve is a continuous curve containing no simple closed curve.

Acyclic continuous curves have been studied by Mazurkiewicz,‡ R. L. Wilder,§ R. L. Moore,‖ and the author.¶ As a result of Theorem 1, it follows that certain internal properties of an acyclic continuous curve which have been proved by the above authors for plane curves, are also possessed by curves in n-dimensional space. However, in the present paper, only plane point sets are considered, unless otherwise stated.

THEOREM 1. *Any acyclic continuous curve lying in n-dimensional space can be put into continuous* (1-1) *correspondence with some plane acyclic continuous curve.*

Mazurkiewicz** states that the above theorem is probably true, but he gives no indication of any attempt to prove it. We shall give here a proof based upon the following definition and lemmas, the truth of which is easily established from the definition.

Definition. A continuous (1-1) correspondence between two point sets M_1 and M_2, is said to be *uniformly continuous*, if given any positive number ϵ, there exists a corresponding positive number δ, such that if A_i and B_i are any two points of M_i at a distance apart less than δ, then the correspond-

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‡ S. Mazurkiewicz, *Un théorème sur les lignes de Jordan*, Fundamenta Mathematicae, vol. 2 (1921), pp. 119–130.

§ R. L. Wilder, *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), pp. 340–377.

‖ R. L. Moore, *Concerning the cut-points of continuous curves and of other closed and connected point sets*, Proceedings of the National Academy of Sciences, vol. 9 (1923), pp. 101–106.

¶ H. M. Gehman, *On extending a continuous* (1-1) *correspondence of two plane continuous curves to a correspondence of their planes*, these Transactions, vol. 28 (1926), pp. 252–265.

** Loc. cit., p. 130. *Note added in proof*: Professor J. R. Kline has informed me that this theorem has been proved by T. Wazewski, Annales de la Société Polonaise de Mathématique, vol. 2 (1923).

ing points A_{i+1} and B_{i+1} of M_{i+1} are at a distance apart less than ϵ, where $i=1, 2$, and all subscripts are reduced modulo 2.*

LEMMA A. *A continuous* (1-1) *correspondence between two closed and bounded point sets is uniformly continuous.*

LEMMA B. *A continuous* (1-1) *correspondence between a closed and an open set cannot be uniformly continuous.*

LEMMA C. *A continuous* (1-1) *correspondence between a bounded and an unbounded set cannot be uniformly continuous.*

LEMMA D. *If two open sets are in uniformly continuous* (1-1) *correspondence, then the correspondence can be extended to the closed sets obtained by adding limit points to the two sets.*

LEMMA E. *If any two sets are in uniformly continuous* (1-1) *correspondence, then if any sequence of points of one of the sets has a sequential limit point* (*not necessarily in the set*), *then the sequence of corresponding points in the other set also has a sequential limit point.*

The set of continuous (1-1) correspondences having the property mentioned in Lemma E (i.e., that a sequence having a sequential limit point always corresponds to a sequence having a sequential limit point) includes as a proper subset the set of uniformly continuous (1-1) correspondences as defined above. If a uniformly continuous (1-1) correspondence had been defined by means of this property, all the lemmas would have remained true, and in addition, the words "*and bounded*" may be omitted from Lemma A. Under either definition, the properties of *boundedness* and *closedness* are invariant properties of a point set under the group of uniformly continuous (1-1) correspondences, which is not the case under those (1-1) correspondences which are merely continuous.

The following example will show that under our given definition of a uniformly continuous (1-1) correspondence, a continuous (1-1) correspondence between two closed *and unbounded* point sets is not necessarily uniformly continuous. Let M_1 be the two lines $y=0$, and $y=1$. Let M_2 be the line $y=0$, and the exponential curve $y=e^x$. Let the correspondence between them be such that each point on $y=0$ corresponds to itself, and each point on $y=1$ corresponds to the point with the same abscissa on $y=e^x$. If ϵ is selected as less than 1, then no matter what δ is selected, there are two points of M_2 (i.e., a point of $y=0$ and a point of $y=e^x$ with the same abscissa),

* Compare the definition of uniform continuity given by W. H. Young and G. C. Young, *The Theory of Sets of Points*, 1906, p. 218.

which are at a distance apart less than δ, and yet the corresponding points of M_1 are at a distance apart equal to 1, and therefore greater than ϵ. Therefore this continuous (1-1) correspondence between two closed and unbounded sets is not uniformly continuous.

The proof of Theorem 1 then proceeds as follows: R. L. Wilder† has shown that an acyclic continuous curve M can be expressed as the sum of a set M^* and a totally disconnected set of limit points of M^*, where M^* is a set which (1) is composed of a sequence of arcs $C_1, C_2, C_3, \cdots$, no two of which have in common a point which is an interior point of both, and (2) is such that if n is any positive integer, $C_1+C_2+\cdots+C_n$ is an acyclic continuous curve which is a proper subset of M, and (3) is such that given any positive number ϵ, there exists a number ρ such that if $n>\rho$, the diameter of C_n is less than ϵ, and the diameter of each tree‡ in $M-(C_1+C_2+\cdots+C_n)$ is less than ϵ. Although Wilder's proof is worded for the case of a plane continuous curve, it is evident that with the necessary changes in wording, his proof will hold also for an acyclic continuous curve lying in a space of any number of dimensions.

Let M_1 be an acyclic continuous curve lying in n-dimensional space, and let M_1 be expressed as described in Wilder's theorem. In a plane S, we can construct an arc D_1 in continuous (1-1) correspondence with C_1, and an arc D_2 such that D_1+D_2 is in continuous (1-1) correspondence with C_1+C_2, preserving the given correspondence between D_1 and C_1, and, in general, an arc D_i such that $D_1+D_2+\cdots+D_i$ is in continuous (1-1) correspondence with $C_1+C_2+\cdots+C_i$, preserving the given correspondence between $D_1+D_2+\cdots+D_{i-1}$ and $C_1+C_2+\cdots+C_{i-1}$. This construction can be performed in such a way that $D_1+D_2+\cdots$ plus limit points of this sequence is an acyclic continuous curve M_2 in S.

By Lemma D, it will be sufficient to prove that $M_1^*=C_1+C_2+\cdots$ and $M_2^*=D_1+D_2+\cdots$ are in uniformly continuous (1-1) correspondence. Suppose that this were not true. Then for some positive number ϵ, there is a sequence of pairs of points X_i, Y_i $(i=1, 2, 3, \cdots)$ of one of the sets, say M_1^*, such that the distance between X_i and Y_i is less than $1/i$, while the distance between the corresponding points X_i' and Y_i' of M_2^* is greater than ϵ. Let us select an integer k so large that the diameter of any tree in $M_2-(D_1+D_2+\cdots+D_k)$ is less than $\epsilon/3$. Then each of the arcs $X_i'Y_i'$ of M_2 contains a subarc $A_i'B_i'$ which lies in $D_1+D_2+\cdots+D_k$, and is of diameter greater than $\epsilon/3$, and therefore contains two points E_i', F_i' whose distance

† Loc. cit., Theorem 15, p. 365.

‡ A *tree* is a *maximal connected subset*. See H. M. Gehman, loc. cit., p. 256.

apart is greater than $\epsilon/3$. Since M_2 was constructed in such a way that the correspondence between $C_1+C_2+\cdots+C_k$ and $D_1+D_2+\cdots+D_k$ is continuous, and therefore, by Lemma A, uniformly continuous, it follows that there exists a constant δ, such that if two points of $C_1+C_2+\cdots+C_k$ are at a distance apart less than δ, the corresponding points of $D_1+D_2+\cdots+D_k$ are at a distance apart less than $\epsilon/3$. Also the set M_1 is uniformly connected im kleinen, and therefore there exists a constant α, such that any two points of M_1 at a distance apart less than α can be joined by an arc in M_1 of diameter less than δ. If we then select an integer i, such that $(1/i)<\alpha$, the diameter of the arc X_iY_i is less than δ. The points E_i, F_i, which are on the arc X_iY_i, are therefore at a distance apart less than δ, and the corresponding points E_i', F_i' are at a distance apart less than $\epsilon/3$, which is contrary to the method of selection of the points E_i', F_i'.

Exactly the same contradiction is obtained if we suppose the points X_i, Y_i to lie in M_2^*. Therefore M_1^* and M_2^* are in uniformly continuous (1-1) correspondence. and Theorem 1 is true.

Definition. If M is a connected point set, and P is a point of M, then if $M-P$ is not connected, P is said to be a *cut point* of M; if $M-P$ is connected, P is said to be a *non-cut point* of M.†

Definition. If M is a continuous curve, and P is a point of M, then P is said to be an *end point* of M, if the maximal connected subset of $M-(A-P)$ containing P consists of P alone, where A is any arc of M having one end point at P.‡ It follows that an end point is always a non-cut point. If M is an acyclic continuous curve, every non-cut point is an end point.

THEOREM 2. *If T, the set of all non-cut points of a bounded continuum M lying in n-dimensional space, is a subset of a closed, totally disconnected subset T' of M, then M is an acyclic continuous curve, the set of whose end points is identical with T.*

Suppose that M were not a continuous curve. In that case, M contains a sequence of mutually exclusive continua W, M_1, M_2, $\cdots$ such that W is the sequential limiting set of the sequence M_1, M_2, $\cdots$, and such that there exists a connected subset of M containing $M_1+M_2+\cdots$ but not containing any points of W.§ If to this connected set we add its limit points,

† R. L. Moore, loc. cit., p. 101.

‡ R. L. Wilder, loc. cit., p. 358. See Theorem 7. For a number of definitions of an end point of a continuous curve which are equivalent to the above, see the author's forthcoming paper *Concerning end points of continuous curves and other continua.*

§ R. L. Wilder, loc. cit., p. 371. See also R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, Bulletin of the American Mathematical Society, vol. 29 (1923), p. 296. We shall refer to this hereafter as *Report.*

the resulting continuum R is a subset of M and contains W. It is well known that if to a connected set, we add any set of its limit points, the resulting set is connected. If then we add to the connected set mentioned above, all its limit points save a point P of W, the resulting set $R-P$ is connected. In other words, every point of W is a non-cut point of R.

Under these conditions, only a countable number of points of W can be cut points of M.† Since the set T' which contains all the non-cut points of M is closed and totally disconnected, there is a subcontinuum of W which contains no points of T', and which therefore consists entirely of cut points of M. But a continuum cannot consist of a countable set of points, and we have therefore arrived at a contradiction by supposing that M is not a continuous curve.

Suppose next that the continuous curve M contains a simple closed curve J. By the theorem referred to above, the cut points of M on J are countable. Since T' is closed and totally disconnected, J contains an arc which contains no points of T', and which therefore consists entirely of cut points of M. But an arc cannot consist of a countable number of points, and we have therefore arrived at a contradiction by supposing that M contains a simple closed curve. The continuum M is therefore an acyclic continuous curve, and therefore the set T of non-cut points is identical with the set of end points of M.

Note however, that if T denotes the set of end points of an acyclic continuous curve, it does not follow that there exists a closed, totally disconnected set T' which contains T. For there exist acyclic continuous curves in which every point is a limit point of the set of end points, and in such a case any closed set which contains T contains the acyclic continuous curve and therefore cannot be totally disconnected.

THEOREM 3. *If T' is a closed and totally disconnected subset of a bounded continuum M in n-dimensional space, then a necessary and sufficient condition that M be an acyclic continuous curve, the set of whose end points is a subset of T', is that M be irreducibly connected about T'.*‡

The condition is necessary, as a bounded continuum is always irreducibly connected about any set which contains its non-cut points, as we have proved in a recent paper.§ In this same paper, we prove that a bounded continuum

† R. L. Moore, *Concerning the cut-points*, etc., loc. cit., Theorem B*, p. 102, and *Report*, Theorem $\bar{D}$, p. 300.

‡ A set M is said to be *irreducibly connected about a set of points* T', if M is connected and contains T', but no proper connected subset of M contains T'.

§ H. M. Gehman, *Concerning irreducibly connected sets and irreducible continua*, Proceedings of the National Academy of Sciences, vol. 12 (1926), pp. 544–547.

is not irreducibly connected about any set which does not contain all its non-cut points, and therefore in proving that the given condition is sufficient it follows that T' contains all the non-cut points of M. Then, by Theorem 2, the continuum M is an acyclic continuous curve, whose end points are a subset of T'. This completes the proof of Theorem 3.

Note that this theorem serves to characterize a certain type of acyclic continuous curve by the condition which Lennes* used to define a simple continuous arc.

R. L. Moore and J. R. Kline† have solved the problem of determining the most general plane point set through which an arc may be passed. It is evident, however, that in the space consisting of a plane continuous curve,‡ their conditions are not sufficient. For if the continuous curve K consists of three arcs, AB, AC, AD, having no points in common save the point A, then the point set consisting of B, C, and D satisfies the hypotheses of their theorem, and yet K contains no arc containing B, C, and D.

In Theorem 4, we shall describe the most general closed point set lying in a plane continuous curve K, through which an acyclic continuous curve lying in K can be passed. In our discussion, it is immaterial whether the continuous curve K be a bounded portion of the plane, or some other type of continuous curve.

LEMMA F. *If N is a closed bounded set consisting of a collection of connected sets (E), each one of which is a maximal connected subset of N, no one of which separates the plane S, and not more than a finite number of which are of diameter greater than any given positive number, then N cannot separate the plane, and if moreover, a point P of a maximal connected subset e of (E) is accessible from $S-e$, then P is accessible from $S-N$.*

Under these hypotheses, the maximal connected subsets in (E) are mutually exclusive and closed. The collection (E) may contain an uncountable collection of these continua, but in that case those which contain more than one point form a countable collection.

If we add to the collection of continua (E), the collection of mutually exclusive continua each consisting of a single point of $S-N$, the resulting collection (G) forms an upper semi-continuous collection of mutually ex-

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* N. J. Lennes, *Curves in non-metrical analysis situs*, American Journal of Mathematics, vol. 33 (1911), p. 308.

† *On the most general plane closed point set through which it is possible to pass a simple continuous arc*, Annals of Mathematics, (2), vol. 20 (1919), pp. 218–223.

‡ This idea has been discussed by R. L. Wilder, loc. cit., p. 341, and by R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), pp. 254–260.

clusive bounded continua, filling up the entire plane S. Under these conditions R. L. Moore* has proved that if each continuum of (G) be considered as a point, the set of elements of (G) is (from the viewpoint of analysis situs) equivalent to the set of points in a plane S'. The set of points in S' corresponding to the elements of (E) is a closed, totally disconnected set of points, N', through which, by the Moore-Kline theorem, an arc may be passed in S'. Any two points of $S'-N'$ can therefore be joined by an arc having no points in common with N'. Each point of S' on this arc corresponds to a single point in S, and therefore any two points of $S-N$ can be joined by an arc having nothing in common with N. The set N therefore does not separate the plane.

Let us now assume that P is a point of a continuum e of (E), which is accessible from $S-e$. Let PQ be an arc joining P to any point Q not in (E), and having only P in common with e. If this arc has points in common with any continuum of (E) of diameter greater than 1, we can enclose each of these continua by a simple closed curve containing no points of N, such that P and Q are in its exterior, such that every point of the simple closed curve and its interior lies at a distance less than $\frac{1}{2}$ from the continuum that it encloses, and such that each simple closed curve of the set is exterior to each of the others.† In the same way, we can enclose each continuum of diameter greater than $\frac{1}{2}$ (but less than 1) that has points in common with a sub-arc of PQ which is exterior to all the simple closed curves of the first set, by a simple closed curve which has the properties of those of the first set, and which in addition is such that every point of the simple closed curve and its interior lies at a distance less than $\frac{1}{4}$ from the continuum that it encloses. Evidently the diameter of each simple closed curve of this second set is less than $\frac{3}{2}$. If we continue this process, we obtain a point set A consisting of the arc PQ and a countable collection of mutually exclusive simple closed curves, each having points in common with the arc PQ, and such that only a finite number of them are of diameter greater than any given positive number. This set A is a continuum which satisfies a set of conditions which are sufficient that every subcontinuum of A be a continuous curve.‡ Therefore the continuum B consisting of the simple closed curves in A and those points

* R. L. Moore, *Concerning upper semi-continuous collections of continua*, these Transactions, vol. 27 (1925), pp. 416–428.

† This is an extension of a theorem due to Zoretti. See R. L. Moore, *Concerning the separation of point sets by curves*, Proceedings of the National Academy of Sciences, vol. 11 (1925), pp. 469–476.

‡ H. M. Gehman, *Some conditions under which a continuum is a continuous curve*, Annals of Mathematics, (2), vol. 27 (1926), pp. 381–384. See especially Theorem 2, p. 382.

of the arc PQ that are not interior to any simple closed curve in A, is a continuous curve that contains P and Q. Therefore B contains an arc PQ.

The arc PQ in B may contain points of N, but if so, the set of points common to N and PQ is totally disconnected. Let $P_1, P_2, P_3, \cdots$ be a sequence of points on PQ which are not points of N, and which are such that the diameter of the arc P_iP is less than $1/i$. The points of N on QP_1 can be covered by a finite set of mutually exclusive simple closed curves of diameter less than 2, having no points in common with N, and such that P and Q are in the exterior of each simple closed curve of the set. The points of N on P_iP_{i+1} can be covered by a similar finite set of simple closed curves of diameter less than $1/i$ $(i=1, 2, 3, \cdots)$. Then by precisely the same argument as in the preceding paragraph, we obtain an arc PQ which has no points in common with N. The point P is therefore accessible from $S-N$, and the truth of Lemma F is established.

THEOREM 4. *A necessary and sufficient condition that a closed subset N of a continuous curve K be a subset of an acyclic continuous curve lying in K, is that every maximal connected subset of N be an acyclic continuous curve, and that not more than a finite number of these maximal connected subsets be of diameter greater than any given positive number.*

The condition is necessary, because every subcontinuum of an acyclic continuous curve is an acyclic continuous curve,* and therefore not more than a finite number of mutually exclusive subcontinua can be of diameter greater than any given positive number.†

To prove that the condition is sufficient, we shall suppose that a continuous curve K contains a closed subset N satisfying the given condition, and shall show how to construct an acyclic continuous curve M lying in K and containing N.

First we shall enclose all those maximal connected subsets of N that are of diameter greater than 1 by a finite collection (C) of mutually exclusive simple closed curves, each of which encloses one maximal connected subset of N of diameter greater than 1, encloses no other maximal connected subset of N of diameter greater than $\frac{1}{2}$, contains no point of N, and is such that every point of the simple closed curve and its interior is at a distance less than 1 from a point of the subset of N of diameter greater than 1, that it encloses.

Let C_1 be a simple closed curve of (C), and let N_1 be the subset of N of diameter greater than 1, enclosed by C_1. Let AB be a maximal arc of

* S. Mazurkiewicz, loc. cit., Lemma, p. 123.

† H. M. Gehman, *Concerning the subsets of a plane continuous curve*, Annals of Mathematics, (2), vol. 27 (1925), pp. 29–46. See Theorem V, p. 39.

N_1 of diameter greater than 1. If A is a limit point of $N-N_1$, we shall put about A as center a circle C_A of diameter less than $\frac{1}{3}$, whose exterior contains B and C_1, and a circle C_A' of diameter half that of C_A. Under the given hypotheses, there are only a finite number of maximal connected subsets of N that have points interior to C_A' and points exterior to C_A. Therefore we can put about A as center a circle C_A'' whose exterior contains every maximal connected subset of $N-N_1$ that contains points exterior to C_A. Since K is connected im kleinen at A, there exists a circle C_A^* about A as center and interior to C_A'', such that every point of K within C_A^* can be joined to A by an arc in K lying within C_A''. Since A is a limit point of $K-N_1$, we can select a point X of $K-N_1$ lying within C_A^* and can select a definite arc XA in K lying within C_A''. Let Y be the first point which this arc has in common with N_1. The arc XY together with those maximal connected subsets of $N-N_1$ that have points in common with XY forms a continuum M, every subcontinuum of which is a continuous curve,† which has only the point Y in common with N_1, and which lies entirely within C_A. The maximal connected subsets of $N-N_1$ that are of diameter >0 and lie in M can be arranged in a sequence $H_1, H_2, H_3, \cdots$, such that given any positive number ϵ, there exists an integer n, such that for $i>n$, H_i is of diameter less than ϵ. Let X_1 and Y_1 be respectively the first and last points that the arc XY has in common with H_1, and let M_1 denote the arc X_1Y_1 of H_1, the arcs XX_1 and Y_1Y of the original arc XY, plus those sets of the sequence $H_2, H_3, H_4, \cdots$ which have points in common with either XX_1 or Y_1Y. Let H_{n_2} be the first member of the sequence $H_2, H_3, H_4, \cdots$ that lies in M_1, and let X_2 and Y_2 be respectively the first and last points that the arc XX_1Y_1Y has in common with H_{n_2}, and let M_2 denote the arc X_2Y_2 of H_{n_2}, the arcs XX_2 and Y_2Y of the arc XX_1Y_1Y, plus those sets of the sequence $H_{n_2+1}, H_{n_2+2}, \cdots$ which have points in common with either XX_2 or Y_2Y. The method of defining $M_3, M_4, \cdots$ is obvious. The common part of the sequence $M, M_1, M_2, \cdots$ is a new arc XY in K having only Y in common with N_1, and lying within C_A, and therefore lying within C_1 and of diameter less than $\frac{1}{3}$. Also from the method in which it has been constructed, the arc XY has the property that if it has two points P, Q in common with a maximal connected subset of N, the arcs PQ of XY and of N are identical. (The arcs $B'Z$, $G'V$, X_iY_i, etc., to be constructed later, are also to have this property.) Since N_1 is a maximal connected subset of N, the arc XY contains a point of $K-N$. Let A' be such a point.

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† H. M. Gehman, *Some conditions under which a continuum is a continuous curve*, loc. cit., Theorem 2, p. 382.

In case A is not a limit point of $N-N_1$, let $A=Y=A'$. In either case, if we add to N_1 the arc $A'Y$ plus all maximal connected subsets of N that have points in common with $A'Y$, we obtain a new acyclic continuous curve N_2 that contains N_1.

In the same way, if the point B is a limit point of $N-N_1$, there is a point B' of $K-N$, and an arc $B'Z$ of K of diameter less than $\frac{1}{3}$, which is interior to C_1 and has no points in common with the arcs $A'Y$ and AY, and has only Z in common with N_2. In case B is not a limit point of $N-N_1$, let $B=B'=Z$. If we add to N_2 the arc $B'Z$ and all maximal connected subsets of N that have points in common with $B'Z$, we obtain a new acyclic continuous curve N_3, containing N_1 and N_2, and such that $A'B'$ is a maximal arc of N_3 whose end points are not limit points of $(N+N_3)-N_3$. The arc $A'B'$ has the arc YZ in common with N_1, and is therefore of diameter greater than $\frac{1}{3}$, because, the diameters of the arcs AY and BZ of N_1 being each less than $\frac{1}{3}$, the diameter of YZ is greater than $\frac{1}{3}$.

By Lemma F, each point of N_3 is accessible from $S-(N+N_3)$. We can therefore join* A' to a point D of C_1 by an arc in S having only A' in common with $N+N_3$, and only D in common with C_1. We can also join B' to a point E of C_1 $(E\neq D)$ by an arc having only B' in common with $N+N_3$, only E in common with C_1, and having no points in common with the arc $A'D$.

If the set $N_3-A'B'$ contains a tree T of diameter greater than $\frac{1}{5}$, it is necessarily a tree of N_1-YZ. Let the foot of the tree be F, and let FG be a maximal arc of $T+F$ of diameter greater than $1/10$. If G is a limit point of $N-N_1$, as before we can construct an arc $G'V$ in K, within a circle about G of diameter less than $1/30$ whose exterior contains C_1 and the arcs DA', $A'B'$, $B'E$, the arc $G'V$ being such that G' is a point of $K-(N+N_3)$, and V is the only point that it has in common with N_3. Since the diameter of $G'V$ is less than $1/30$, the diameter of the arc VF of N_3 is greater than $2/30$.

If G is not a limit point of $N-N_1$, let $G=G'=V$. Let N_3 plus $G'V$ plus all maximal connected subsets of N that have points in common with $G'V$ be denoted by N_4, which is also an acyclic continuous curve. As before, $N+N_4$ satisfies the hypotheses of Lemma F, and therefore the point G' can be joined to a point H of C_1 $(D\neq H\neq E)$ by an arc having only G' in common with $N+N_4$, only H in common with C_1, and having no points in common with either $A'D$ or $B'E$.

If the set $N_4-(A'B'+FG')$ contains a tree of diameter greater than $\frac{1}{5}$, it is necessarily a tree of $N_1-(YZ+FV)$, and as we have done before we

* The construction from this point follows in the main that used in the proof of Theorem 1 of my paper *On extending a continuous* (1-1) *correspondence*, etc., loc. cit.

can construct an arc within C_1 from a point of C_1 to a point of $K-(N+N_4)$ or a point of N_4 (depending upon whether the end point of N_4 first chosen is or is not a limit point of $N-N_1$), and obtain as before an acyclic continuous curve N_5 which contains N_4 as a subset.

After a finite number k of steps, $N_k-(A'B'+FG'+\cdots)$ will contain no tree of diameter greater than $\frac{1}{5}$, otherwise N_1 would contain an infinite set of mutually exclusive arcs of diameter greater than $2/30$ which is impossible.

Therefore, after the kth step, the interior of C_1 has been expressed as the sum of a finite number of domains, plus boundary points of these domains, where the boundary of each domain is a simple closed curve consisting of an arc of N_k, and an arc having only its end points in common with $N+N_k$. The arc of N_k lying in the boundary is a maximal arc of N_k, and its end points are not limit points of $(N+N_k)-N_k$.

Let J denote one of these simple closed curves, PQ the arc of N_k in J, N' the points of N_k contained in and enclosed by J. We can select a finite set of points $P_1, P_2, \cdots, P_n$ on PQ, such that (a) $P_1=P$, (b) $P_n=Q$, (c) P_i precedes P_{i+1} on PQ, for $i=1, 2, \cdots, n-1$, (d) the diameter of each arc P_iP_{i+1} is less than $\frac{1}{5}$, (e) no tree in $N'-PQ$ has its foot at any of the points P_i.

If P_i is a limit point of points of $(N+N_k)-N_k$ interior to J, then as before, there is a point X_i of $K-(N+N_k)$, and an arc X_iY_i in K having only Y_i in common with N_k, interior to J save possibly for Y_i, of diameter less than $\frac{1}{5}$, such that no two arcs of the set X_iY_i have points in common or contain points of the same maximal connected subset of $(N+N_k)-N_k$, and such that the arc Y_iP_i of N_k does not contain any other points of the sets $P_1, P_2, \cdots, P_n$ and $Y_1, Y_2, \cdots, Y_n$. If P_i is not a limit point of $(N+N_k)-N_k$, then $P_i=X_i=Y_i$. Let N' plus the arcs $X_1Y_1, X_2Y_2, \cdots, X_nY_n$ plus all maximal connected subsets of N that have points in common with one of these arcs, be denoted by N''.

As in the paper referred to in the previous footnote, there exists a set of arcs $X_1X_2, X_2X_3, \cdots, X_{n-1}X_n$ in S, which have no points in common save end points of consecutive arcs of the sequence, are interior to J save possibly for end points, are of diameter less than 2, and have no points in common with $N+N_k+N''$ save their end points. These arcs form with the corresponding arcs of N'' a set of simple closed curves $J_1, J_2, \cdots, J_{n-1}$ which contain or enclose all points of N''. The diameter of the arc X_iX_{i+1} of N'' is less than 1, and therefore the diameter of each of the simple closed curves J_i is less than 3.

If the above construction is made in each of the simple closed curves J, we have a finite set of simple closed curves (similar to J_i) each of diameter

less than 3, which contain or enclose all points of N_k, and are such that each is composed of an arc in K and an arc no point of which is a limit point of points of N. No two of these simple closed curves have an interior point in common.

We shall now make the same construction in each of the simple closed curves of the set (C) that we have made within C_1. Let (D) denote the collection of all the simple closed curves similar to J_i. If any maximal connected subsets of N are exterior to (D), we shall cover each such set by a simple closed curve containing no points of N or of (D), enclosing no points of (D), and such that every point of the simple closed curve and its interior is at a distance less than 1 from a point of the subset of N that it encloses. Since any subset of N exterior to (D) is of diameter less than 1, it follows that each simple closed curve of this set is of diameter less than 3. If this set is infinite, we can select from it a finite subset (E) which encloses all points of N exterior to (D). If any two simple closed curves of (E) have points in common, the collection (E) may be replaced by a finite collection (F) of simple closed curves enclosing all points of N exterior to (D), each simple closed curve being of diameter less than 3, and having the additional property that no two have a point or an interior point in common.*

Let R denote one of the simple closed curves in $(D)+(F)$. Any maximal connected subset of K contained in R plus its interior is a continuous curve, and no more than a finite number of these continuous curves are of diameter greater than any given positive number.† If R is in (F), then since there is some constant d which is less than the distance from a point of N to a point of R, it follows that the points of N in the interior of R lie in a finite number of mutually exclusive continuous curves which are subsets of K. Similarly, if R is in (D), then, since there is some constant d' which is less than the distance from a point of N to the arc that has no points in common with N (and since the other arc of (R) lies in K), it follows that the points of N in the interior of R and on R, lie in a finite number of mutually exclusive continuous curves which are subsets of K.

Therefore all points of N lie in a finite collection (G) of continuous curves which are subsets of K and which are of diameter less than 3. Let G_1, G_2, $\cdots$, G_a be the finite set of maximal connected subsets of (G), and let P_i be a point in G_i. We can join P_1 to P_2 by an arc in K, and on this arc, we

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* Moore and Kline, loc. cit., Lemma 1, p. 219.

† H. M. Gehman, *Concerning the subsets of a plane continuous curve*, loc. cit., Lemma B, p. 34. See also Theorem 6 of H. M. Gehman, *Some relations between a continuous curve and its subsets*, Annals of Mathematics, (2), vol. 28 (1927), pp. 103–111.

can select all possible arcs AB, such that A is in G_1 and B is in one of the sets $G_2, G_3, \cdots, G_a$, and AB has no other points in common with any of the sets G_i. If there is more than one arc AB such that B is a point of G_i (for a fixed value of i), then select one definite arc AB from G_1 to G_i. The set G_1 plus the arcs AB plus the sets of $G_2, G_3, \cdots, G_a$ that contain one of the points B, we shall call H_1. The other sets of $G_2, G_3, \cdots, G_a$ we shall call $H_2, H_3, \cdots, H_b$. Note that $b \leqq a-1$. Let Q_i be a point of H_i, and construct an arc in K from Q_1 to Q_2, and then proceed as with the arc P_1P_2.

After a finite number of steps we obtain a continuous curve K_1 which is a subset of K, contains N, contains no point exterior to the simple closed curves in $(D)+(F)$ except points of the arcs AB, and has the property that there is only one "path" composed of arcs AB and sets G_i, joining any two points of K_1.

If we now consider each member G_i of the finite collection (G) of continuous curves of diameter less than 3 which are subsets of K and contain all of N, we can perform the same type of construction with G_i within and on the simple closed curve that encloses and contains G_i, as we have just performed with K, in such a way as finally to cause all the points of N that are contained in G_i to lie in a finite number of continuous curves that are subsets of G_i (and therefore of K), and are of diameter less than 1. Having done that with each member of (G), we can construct arcs AB in K_1, so as to obtain a continuous curve K_2 which is a subset of K_1, contains N, contains no point save points of the arcs AB exterior to the finite set of simple closed curves that enclose all of N, and has the property that there is only one "path" joining any two points of K_2.

Continuing this process, the set of points common to $K, K_1, K_2, \cdots$ is a bounded continuum M, which is a subset of K and contains N. We shall now show that M is an acyclic continuous curve.

Sierpinski* has shown that a bounded continuum M is a continuous curve if M can be expressed as the sum of a finite collection of continua each of diameter less than a preassigned positive number. Let us then select a positive number d. There is an integer n such that $1/3^n < d$. The set K_{n+2} consists of a finite collection of continuous curves each of diameter less than $1/3^n$ plus a finite collection of arcs, each of which may be expressed as the sum of a finite collection of arcs each of diameter less than $1/3^n$. The set M consists of these arcs plus some points of the continuous curves, i.e., for each continuous curve in K_{n+2}, M contains only the points common to the

* W. Sierpinski, *Sur une condition pour qu'un continu soit une courbe jordanienne*, Fundamenta Mathematicae, vol. 1 (1920), pp. 44–60.

continuous curve and all the continua of the sequence $K, K_1, K_2, \cdots$. Since this common part is always a continuum, the part in M is a continuum which is of diameter less than $1/3^n$, and therefore M is a continuous curve.

Suppose that M contains a simple closed curve J. Since N contains no simple close curves, J contains a point P which is not a point of N. If the distance from P to a point of N is greater than $1/3^m$, it is evident that P cannot be a point of a continuous curve in K_{m+2}, and therefore P is a point of an arc AB of K_{m+2}. If X and Y are two points on this arc, such that the arc XPY contains no point of N, then since J lies in K_{m+2} there are evidently two "paths" joining X to Y, contrary to our method of constructing the sets K_i. Therefore M is an acyclic continuous curve.

THEOREM 5. *A necessary and sufficient condition that a bounded continuum K be a continuous curve is that every closed, totally disconnected subset T' of K be a subset of some subcontinuum M of K which is irreducibly connected about T'.*

In view of Theorem 3, Theorem 5 is equivalent to the following theorem:

THEOREM 5′. *A necessary and sufficient condition that a bounded continuum K be a continuous curve is that every closed, totally disconnected subset T' of K be a subset of an acyclic continuous curve M in K, the set of whose end points is a subset of T'.*

The condition is necessary, because T' satisfies the conditions of Theorem 4, and therefore an acyclic continuous curve M can be constructed in K containing T', by the method used in the proof of that theorem. As was pointed out in the final paragraph of the proof of Theorem 4, a point of M that is not a point of T' is a point of an arc introduced at a certain step, and is therefore a cut point of each of the continuous curves K_i after that step, and also a cut point of M. Therefore the set of end points of M is a subset of T'.

The condition is sufficient. For if K were not a continuous curve, it would contain a sequence of continua $W, M_1, M_2, \cdots$, having the properties mentioned in the proof of Theorem 2, and each having at least one point in common with each of two concentric circles, C_1 and C_2, and lying entirely in the point set H composed of the circles and all points of the plane between them, each set (save possibly W) being a maximal connected subset of K in H.

Let P be a point of W lying between C_1 and C_2, and $P_1, P_2, \cdots$ a sequence of points such that P_i is a point of M_i, and such that P is the sequential limit of the sequence. Let $P+P_1+P_2+\cdots$ be T'. By hypothesis, K contains a continuous curve M containing T'. Therefore all but a finite

number of the points of T' can be joined to P by an arc in K and in H, which contradicts our supposition that each set in the original sequence (except possibly W) was a maximal connected subset of K in H. Therefore K is a continuous curve, and the condition is sufficient.

THEOREM 6. *Every closed, bounded, totally disconnected point set is identical with the set of end points of some acyclic continuous curve.*

Let T' be a closed, bounded, totally disconnected set. We shall show how to construct an acyclic continuous curve, the set of whose end points is identical with T'.

An arc AB may be passed through T', by the Moore-Kline theorem, in such a way that its end points are points of T'. Then a set of points belonging to $AB-T'$ can be selected which divide AB into a finite number of intervals, $I_1, I_2, \cdots, I_n$, each of diameter less than 1. If I_r contains an infinite number of points of T', we shall denote the first point of T' in I_r by A_r, and the last point by D_r. We shall denote by B_r and C_r two points of T' on the arc A_rD_r, such that the arc B_rC_r contains no other points of T', and such that the four points occur on I_r in the order $A_rB_rC_rD_r$.

Let P be a point of the plane not on AB. Let us construct arcs PA and PB having only P in common, and having only A and B respectively in common with AB, thus forming a simple closed curve which we shall denote by J. Then construct a finite set of arcs, each one interior to J save for its end points, and no pair having any points in common save P, by joining P to each point of $T'-(A+B)$ in those intervals of AB that contain only a finite number of points of T', and by joining P to each of the points A_r and D_r in the remaining intervals. In case I_1 contains an infinite number of points of T', let the arc PA_1 be the arc PA of J; in case I_n contains an infinite number of points of T', let PD_n be the arc PB of J.

For each value of r for which the points A_r, B_r, C_r, D_r have been defined, we next make the following construction. We enclose the arc A_rB_r by a simple closed curve J_r, whose exterior contains the arc C_rD_r and all the arcs that have been constructed from P to points of AB (excepting of course, the arc PA_r), and which is such that every point of J_r and its interior is at a distance less than 1 from a point of A_rB_r. The diameter of J_r is therefore less than 3. If X is the first point that A_rP has in common with J_r, then B_r can be joined to any point E_r of $A_rX(A_r\neq E_r\neq X)$ by an arc interior to J_r, interior to the simple closed curve PA_rD_r save for its end points, having only E_r in common with A_rP and only B_r in common with AB. The arcs B_rE_r, E_rA_r, and A_rB_r are each of diameter less than 3, and form a simple closed curve which is also of diameter less than 3.

By a similar construction, we obtain an arc C_rF_r, where F_r is a point of $D_rP(D_r \neq F_r \neq P)$, such that the arc has only C_r in common with AB, only F_r in common with PD_r, has no points in common with any other of the arcs joining P to points of AB, has no points in common with B_rE_r, is interior to PA_rD_r save for its end points, is of diameter less than 3, and forms with the arcs F_rD_r and C_rD_r a simple closed curve of diameter less than 3.

Let us denote by K_1 the set consisting of all arcs joining P to points of AB, plus the set of simple closed curves $A_rB_rE_r$ and $C_rD_rF_r$ and their interiors. This set has all the properties of the set K_1 obtained in the proof of Theorem 4. If we continue the same method of construction within each of the simple closed curves $A_rB_rE_r$, $C_rD_rF_r$ that we have performed within J, replacing P by E_r(or F_r), A by A_r(or C_r), and B by B_r(or D_r), dividing the arc A_rB_r (or C_rD_r) into intervals of diameter less than $\frac{1}{3}$, we obtain a set K_2 which is a subset of K_1, and has all the properties of the set K_2 obtained in the proof of Theorem 4.

Continuing this process, the set of points common to $K_1, K_2, \cdots$ is an acyclic continuous curve M, as was shown in the proof of Theorem 4. It is evident that M consists of a connected set consisting of a countable infinity of arcs obtained by the process outlined above, plus limit points of these arcs. The limit points are points of T' which are not end points of arcs of the countable set, and all such points are non-cut points of M. Also the points of T' which are end points of arcs are non-cut points of M, and therefore every point of T' is a non-cut point (and therefore an end point) of M.

The points of $M-T'$ are not points of AB, and are cut points of M, as was shown in the proof of Theorem 4. Therefore the set T' is identical with the set of end points of M.

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ON A GENERAL FORMULA IN THE THEORY OF TCHEBYCHEFF POLYNOMIALS AND ITS APPLICATIONS*

BY

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Introduction. Given a *non-negative function* $p(x)$, defined on a certain interval (a, b), *finite or infinite*, the notations $p(x)-(C)$, $p(x)-(B)$, $p(x)-(I)$, $p(x)-(S)$ will signify respectively $(\alpha): \int_a^b p(x)x^i dx$ exists for $i=0, 1, \cdots$, with $\int_a^b p(x)dx>0$; $(\beta): p(x)$ is bounded; $(\gamma): p(x)$ is (L) integrable; $(\delta): (a, b)$ being finite, (α) is satisfied with $(\log p(x))/((x-a)\cdot(b-x))^{1/2}-(I)$.

The object of this paper is to solve two general problems as follows:

PROBLEM A. *Given an interval (a, b), finite or infinite, a function $q(x)-(C)$ defined on (a, b), certain real constants: $M(>0)$, $\alpha_i(i=0, 1, 2, \cdots, n)$, find the maximum actually attained of the absolute value of the linear expression*

$$\omega(G_n) \equiv \sum_0^n{}_i\alpha_i g_i \tag{1}$$

for all polynomials $G_n(x)=\sum_0^n g_i x^i$ of degree $\leqq n$ satisfying the inequality

$$I(q\,;G_n\,;a,b) \equiv \int_a^b q(x)G_n^2(x)dx \leqq M^2.\dagger \tag{I}$$

PROBLEM A'. *Find the upper limit of $\omega(G_n)$ as defined in (A), for all polynomials $G_n(x)$ such that*

$$E(q\,;G_n\,;a,b) \equiv |\,q(x)G_n(x)\,| \leqq M \qquad (a \leqq x \leqq b),\dagger \tag{II}$$

$q(x)$ arbitrarily defined on (a, b).

These problems arise in many questions, due to the general character of the α_i and $q(x)$.

We give, first, using Tchebycheff polynomials, the solution of (A), (A'), and some general applications of the formulas thus established (§§ 1–4). In order to obtain further applications, we give some properties of certain

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† Equality in (I), (II) being attained by certain $G_n(x)$. We agree to denote by $G_n(x)\equiv\sum_0^n g_i x^i$ an *arbitrary* polynomial of degree $\leqq n$(subjected, in some cases, to additional conditions).

classes of Tchebycheff polynomials (§§ 5–7), which enable us, specifying the α_i and $q(x)$ above, to derive numerous results concerning polynomials in general, and Tchebycheff polynomials in particular (§§ 8–10) on a finite, as well as an infinite interval.

1. **Solution of Problem** A. We proceed, first, to *express* $\omega(G_n)$ *in form of a definite integral.* For this purpose we introduce an auxiliary function $\mathcal{P}(x)$ by means of the relations

$$\int_a^b q(x)\mathcal{P}(x)x^i dx = \alpha_i \qquad (i = 0, 1, \cdots, n). \tag{2}$$

To find $\mathcal{P}(x)$, we consider the system

$$\begin{aligned} \phi_n(q\,;a,b\,;x) &= a_n(q\,;a,b)x^n + \cdots + a_{ni}(q\,;a,b)x^i + \cdots \\ &= a_n[x^n - S_n x^{n-1} + d_{n,n-2}x^{n-2} + \cdots] \quad (n = 0,1,2,\cdots; a_n > 0)^* \end{aligned} \tag{3}$$

of Tchebycheff polynomials corresponding to (a, b) with the "characteristic function" $q(x)$. They are determined *uniquely* by means of the relations

$$\begin{aligned} \int_a^b q(x)\phi_m(x)\phi_n(x)dx &= 0 \quad (m \neq n), \\ &= 1 \quad (m = n). \end{aligned} \tag{4}$$

The set of equations (2) gives, using (1), (4),

$$\int_a^b q(x)\mathcal{P}(x)\phi_i(x)dx = \omega[\phi_i(q\,;x)] \equiv \omega(\phi_i) \quad (i = 0, 1, \cdots, n). \tag{5}$$

Thus, we get, as solution of (2), the *polynomial*†

$$\mathcal{P}(x) = \sum_0^n {}_i\,\omega(\phi_i)\phi_i(x), \tag{6}$$

with

$$\int_a^b q(x)\mathcal{P}^2(x)dx = \sum_0^n {}_i\,\omega^2(\phi_i),$$

which gives for $\omega(G_n)$ the *fundamental* formula

$$\omega(G_n) = \int_a^b q(x)\mathcal{P}(x)G_n(x)dx. \tag{7}$$

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* When convenient, we shall use also abbreviated notations: $\phi_n(q;x)$, $\phi_n(x)$, $a_n(q)$, a_n, $\cdots$.

† A more general solution of (2) *will not better* the results below.

Using Schwartz's inequality, we get

$$\text{(III)} \qquad |\,\omega(G_n)\,| \leqq \left(\int_a^b q(x)G_n^2(x)dx\right)^{1/2}\left(\int_a^b q(x)\mathcal{P}^2(x)dx\right)^{1/2},$$

$$\text{(IV)} \qquad \frac{\omega^2(G_n)}{I(q\,;G_n\,;a,b)} \leqq \sum_0^n{}_i\,\omega^2[\phi_i(q\,;a,b\,;x)],$$

$$|\,\omega(G_n)\,| \leqq M\left(\sum_0^n{}_i\,\omega^2[\phi_i(q\,;x)]\right)^{1/2}.$$

Formulas (III)–(IV) give the required solution:

$$\text{(8)} \qquad \max \frac{\omega^2(G_n)}{I(q\,;G_n\,;a,b)} = \sum_0^n{}_i\,\omega^2[\phi_i(q\,;a,b\,;x)].$$

The maxima above are attained if and only if

$$\text{(9)} \qquad \begin{aligned} &\text{in (8):} \quad G_n(x) = c\sum_0^n{}_i\,\omega(\phi_i)\phi_i(q\,;x) \qquad (c = \text{const.})\,; \\ &\text{in (IV):} \quad G_n(x) = \frac{M}{\left(\sum_0^n{}_i\,\omega^2(\phi_i)\right)^{1/2}}\sum_0^n{}_i\,\omega(\phi_i)\phi_i(q\,;x). \end{aligned}$$

2. **Another form of the solution of (A). Solution of Problem (A').** Assume there exist

(10) in Problem $A: r(x)-(B)$, with $p(x)\equiv q(x)r(x)-(C)$ on (a, b);

(11) in Problem $A': r(x)-(I)$ with $p(x)\equiv q^2(x)r(x)-(C)$ on (a, b).

We have then, correspondingly,

$$\int_a^b p(x)G_n^2(x)dx \leqq r_{\max}\int_a^b q(x)G_n^2(x)dx \leqq M^2 r_{\max}{}^* \qquad \text{(see I)},$$

$$\int_a^b p(x)G_n^2(x)dx \leqq M^2\int_a^b r(x)dx \qquad \text{(see II)}.$$

Thus we come again to (A), and (III) gives

$$\text{(V)} \qquad \begin{aligned} &|\,\omega(G_n)\,| \leqq RM\left(\sum_0^n{}_i\,\omega^2[\phi_i(p;x)]\right)^{1/2} \quad (p(x)\text{ defined in (10), (11)}), \\ &R^2 = r_{\max}\,[\text{in }(A)],\ \int_a^b r(x)dx\,[\text{in }(A')]. \end{aligned}$$

* $f_{\max}$, $f_{\min}$ denote, in general, max $|f(x)|$, min $|f(x)|$ (or upper and lower limits respectively) in (a, b).

Formulas (III)–(V) *give the solution of problems* A, A' *for any interval* (a, b) *with arbitrary* $q(x)$ (*as defined in* (10), (11), *respectively*) *and* α_i.

They yield directly a *general minimum-property of Tchebycheff polynomials*:

$$\text{(VI)} \qquad \min \frac{\int_a^b q(x) G_n^2(x)\,dx}{\omega^2(G_n)} = \frac{1}{\sum_0^n {}_i\, \omega^2(\phi_i)}$$

(minimum attained for $G_n(x) = c\sum_{i=0}^n \omega(\phi_i)\phi_i(x)$ only; $c = \text{const.}$);

$$\text{(VII)} \qquad \begin{aligned} \sum_0^n {}_i\, \omega^2[\phi_i(p\,;\,x)] &\geqq \frac{\omega^2(G_n)}{r_{\max} I(q\,;\,G_n\,;\,a, b)} \qquad (p(x) \equiv q(x) r(x)), \\ \sum_0^n {}_i\, \omega^2[\phi_i(p\,;\,x)] &\geqq \frac{\omega^2(G_n)}{\int_a^b r(x)\,dx\; E(q\,;\,G_n\,;\,a, b)} \qquad (p(x) \equiv q^2(x) r(x)). \end{aligned}$$

We proceed now to derive from (III)–(VII), by specifying the α_i, some important properties pertaining: (a) to polynomials in general; (b) to Tchebycheff polynomials.

3. **Polynomials in general.** (i). For any (a, b) and $q(x) - (C)$, we get, from (III)–(VII), taking

$$\text{(12)} \qquad \begin{aligned} \omega(G_n) &\equiv G_n^{(k)}(z) \qquad (k \geqq 0,\ z \text{ arbitrary}), \\ \omega(G_n) &\equiv \int_{a'}^{b'} f(x) G_n^{(k)}(x)\,dx \\ &\qquad (k \geqq 0,\ a', b' \text{ arbitrary}\,;\ f(x) - (I) \text{ on } (a', b'))\,; \end{aligned}$$

$$\text{(13)} \qquad \begin{aligned} [G_n^{(k)}(z)]^2 &\leqq I(q\,;\,G_n\,;\,a, b) \sum_k^n {}_i\, [\phi_i^{(k)}(z)]^2, \\ \left[\int_{a'}^{b'} f(x) G_n^{(k)}(x)\,dx\right]^2 &\leqq I(q\,;\,G_n\,;\,a, b) \sum_k^n {}_i \left[\int_{a'}^{b'} f(x)\phi_i^{(k)}(q\,;\,x)\,dx\right]^2 \end{aligned}$$

(the equality holds if and only if $G_n(x) = c\sum_{i=0}^n \omega(\phi_i)\phi_i(x)$, $c = \text{const.}$);

$$\text{(14)} \qquad \begin{aligned} [G_n^{(k)}(z)]^2 &\leqq r_{\max} I(q\,;\,G_n\,;\,a, b) \sum_k^n {}_i\, [\phi_i^{(k)}(p\,;\,z)]^2 \\ &\qquad (p(x) \equiv q(x) r(x)), \\ \left[\int_{a'}^{b'} f(x) G_n^{(k)}(x)\,dx\right]^2 &\leqq r_{\max} I(q\,;\,G_n\,;\,a, b) \sum_k^n {}_i \left[\int_{a'}^{b'} f(x)\phi_i^{(k)}(p\,;\,x)\,dx\right]^2; \end{aligned}$$

$$
(15)\quad \begin{gathered}
[G_n^{(k)}(z)]^2 \leqq \int_a^b r(x)dxE(q\,;G_n\,;a,b)\ \sum_k^n{}_i[\phi_i^{(k)}(p\,;z)]^2 \\
(p(x)\equiv q^2(x)r(x)), \\
\left[\int_{a'}^{b'} f(x)G_n^{(k)}(x)dx\right]^2 \leqq \int_a^b r(x)dx\sum_k^n{}_i\left[\int_{a'}^{b'} f(x)\phi_i^{(k)}(p\,;x)dx\right]^2.
\end{gathered}
$$

The relations (13) lead to

THEOREM I. *The integral*

$$
\int_a^b p(x)\left(\sum_0^n{}_i g_i(x-z)^i\right)^2 dx \equiv \int_a^b p(x)G_n^2(x)dx,
$$

where the polynomial $G_n(x)$ *is subject only to the condition* $g_k=1$ *for a specified value of* k, *is minimized by the single polynomial* $k!\sum_{i=k}^n\phi_i^{(k)}(p;z)\phi_i^{(k)}(p;x)\div\sum_{i=k}^n[\phi_i^{(k)}(p\,;z)]^2$. *The required minimum is* $k!\div\sum_{i=k}^n[\phi_i^{(k)}(p\,;z)]^2$. *Here the interval* (a,b), $p(x)-(C)$, z *and the integer* $k(n\geqq k\geqq 0)$ *are arbitrary.*

The two particular cases $k=n$, 0 give known results:

$$
(16)\quad \begin{gathered}
\min\int_a^b p(x)(x^n+\cdots)^2dx = \frac{1}{a_n^2(p\,;a,b)}\,;^* \\
\min\int_a^b p(x)[1+g_1(x-z)+\cdots+g_n(x-z)^n]^2dx = \frac{1}{\sum_0^n{}_i[\phi_i(p\,;z)]^2}.
\end{gathered}
$$

(ii). Consider (13) with $z=0$. We get, with the abbreviated notation,

$$
(17)\quad K_n^{(k)}(q\,;a,b\,;x)\equiv\sum_k^n{}_i[\phi_i^{(k)}(q\,;a,b\,;x)]^2 \qquad (k\geqq 0\,;K_n^{(0)}\equiv K_n),
$$

$$
(18)\quad |g_k|\leqq[I(q\,;G_n\,;a,b)]^{1/2}\left(\sum_k^n{}_i a_{ik}^2(q\,;a,b)\right)^{1/2} \qquad (k\geqq 0\,;a_{ii}\equiv a_i)
$$

(the equality holds if and only if $G_n(x)=c\sum_{i=k}^n\phi_i^{(k)}(q\,;0)\phi_i(q\,;x)$, $c=$const.);

$$
(18\text{ bis})\qquad |g_k|\leqq M\left(\sum_k^n{}_i a_{ik}^2(q\,;a,b)\right)^{1/2},
$$

for all polynomials $G_n(x)$ such that $I(q\,;G_n\,;a,b)\leqq M^2$ (the equality holds for the polynomial

$$
\frac{M\ \sum_{i=k}^n\phi_i^{(k)}(q\,;a,b\,;0)\phi_i^{(k)}(q\,;a,b\,;x)}{[K_n^{(k)}(q\,;a,b\,;0)]^{1/2}}
$$

* $a_n^2(p)$ and $\sum_{i=0}^n\phi_i^2(p;x)$ are of great importance in the theory of Tchebycheff polynomials and its applications.

only);

$$|g_k| \leqq [I(q\,;G_n\,;a,b)r_{\max}]^{1/2}\left(\sum_k^n a_{ik}^2(p\,;a,b)\right)^{1/2}$$

(19) $$[p(x) \equiv q(x)r(x)],$$

$$|g_k| \leqq E(q\,;G_n\,;a,b)\left(\int_a^b r(x)dx\right)^{1/2}\left(\sum_k^n a_{ik}^2(p\,;a,b)\right)^{1/2}$$

$$[p(x) \equiv q^2(x)r(x)].$$

Formulas (13)–(15), (18), (19) *give for an arbitrary* $G_n(x)$ *the upper limits* (*or maxima*) *of its derivatives of different orders at any point, also of any of its coefficients, if we know the maximum of* $\int_a^b q(x)G_n^2(x)dx$ *or of* $|q(x)G_n(x)|$ *on* (a, b), $q(x)$ *satisfying certain general conditions,* (a, b) *being finite or infinite.*

(iii). Our formulas lead also to

THEOREM II. *The relation* $\omega(G_n) \equiv \sum_{i=0}^n \alpha_i g_i = a(\neq 0)$ *implies*

$$\int_a^b q(x)G_n^2(x)dx \geqq \frac{a^2}{\sum_0^n \omega^2[\phi_i(q\,;x)]}$$

for arbitrary α_i *and* $q(x)-(C)$, *equality being attained for the polynomial* $a\sum_{i=0}^n \omega(\phi_i)\phi_i(x) \div (\sum_{i=0}^n \omega^2(\phi_i))$ *only.*

4. **Tchebycheff polynomials in general.** (i). Since (III)–(IV) give the maxima, actually attained, and (V) gives in general the upper limits of the same quantities, we get

(20)
$$\sum_0^n \omega^2[\phi_i(q\,;a,b\,;x)] \leqq r_{\max} \sum_0^n \omega^2[\phi_i(qr\,;a,b\,;x)],$$

$$\sum_0^n \omega^2[\phi_i(q\,;a,b\,;x)] \leqq \sum_0^n \omega^2[\phi_i(q\,;a',b'\,;x)]\ (a \leqq a' < b' \leqq b),$$

for any interval (a, b) *and for arbitrary* $q(x)-C$, $r(x)-B$, α_i. In particular (with $\alpha_n = 1$, $\alpha_{n-1} = \cdots = \alpha_0 = 0$)

(21)
$$a_n^2(q\,;a,b) \leqq r_{\max}a_n^2(qr\,;a,b),^*$$

$$a_n^2(q\,;a,b) \leqq a_n^2(q\,;a',b') \qquad (a \leqq a' < b' \leqq b).$$

---

* Cf. my Note, *Sur quelques propriétés des polynomes de Tchebycheff*, Comptes Rendus, vol. 166 (1918), pp. 28–31; p. 29.

(ii). The relation (VI) gives that $p_2(x) \geqq p(x) \geqq p_1(x)$ in (a, b) implies

$$\sum_0^n {}_i\omega^2[\phi_i(p_2\,;\,x)] \leqq \sum_0^n {}_i\omega^2[\phi_i(p\,;\,x)] \leqq \sum_0^n {}_i\omega^2[\phi_i(p_1\,;\,x)] \tag{22}$$

(in particular) $K_n^{(k)}(p_2\,;\,z) \leqq K_n^{(k)}(p\,;\,z) \leqq K_n^{(k)}(p_1\,;\,z)$ ($k \geqq 0$ and z arbitrary).

(iii). Let (a, b) *be finite*. We get, taking into consideration (13), (18), (19) with

$$k = 0,\, G_n(x) = \cos n \arc\cos\left(\frac{2x-a-b}{b-a}\right) \equiv \sum_0^n {}_k B_k x^k,$$

$$\sum_k^n {}_i a_{ik}^2(q\,;\,a,b) > \frac{B_k^2}{\int_a^b q(x)dx} \qquad (k \geqq 0\,;\, a_{ii} \equiv a_i),$$

(23)

$$\sum_0^n {}_i\phi_i^2(q\,;\,z)$$

$$> \left\{\frac{\{2z-a-b+2[(z-a)(z-b)]^{1/2}\}^n + \{2z-a-b-2[(z-a)(z-b)]^{1/2}\}^n}{2(b-a)^n\left(\int_a^b q(x)dx\right)^{1/2}}\right\}^2$$

$(z < a$ or $> b)$.

Formula (23) *holds for any characteristic function* $q(x)$. For $k = n$ we get the known important inequality (established by the writer in his thesis)

$$a_n(q\,;\,a,b) > \frac{2^{2n-1}}{(b-a)^n\left(\int_a^b q(x)dx\right)^{1/2}}\,. \tag{24}$$

5. **Some special classes of Tchebycheff polynomials.** The results thus far obtained are very general, due to the general nature of $p(x)$, $q(x)$, $r(x)$. By specifying in the formulas above the nature of these functions, further and more concrete results will be obtained. In case of a *finite interval* (reduced, without loss of generality, to $(-1, 1)$) we first take $p(x)-(S)$ (see Introduction) and then, specifying further, we make use of the most important class—the *polynomials of Jacobi* :*

$$p(x) = (1+x)^{\alpha-1}(1-x)^{\beta-1} \qquad (\alpha, \beta > 0),$$

$$\phi_n(x) = \frac{(-1)^n(1+x)^{1-\alpha}(1-x)^{1-\beta}a_n}{(\alpha+\beta+n-1)\cdots(\alpha+\beta+2n-2)}\frac{d^n}{dx^n}[(1+x)^{\alpha+n-1}(1-x)^{\beta+n-1}],$$

(25)

$$a_n = 2^{n+(\alpha+\beta)/2-1}\left(\frac{1}{\pi}\right)^{1/2}[1+o(1)],\; a_{n,n-1} = -a_n\left(\frac{\alpha-\beta}{2}+o(1)\right), \cdots.$$

* Cf. C. Possé, *Sur quelques applications des fractions continues algébriques*, St. Petersburg, 1886, pp. 1–172; pp. 48–65.

*Special cases.** For $\alpha=\beta=\frac{1}{2}$, we have the *trigonometric polynomials*:

$$\phi_n(x) = \left(\frac{2}{\pi}\right)^{1/2} \cos n \arc\cos x \tag{26}$$

$$= \left(\frac{2}{\pi}\right)^{1/2} \frac{[x+(x^2-1)^{1/2}]^n + [x-(x^2-1)^{1/2}]^n}{2} \qquad (n \geqq 1),$$

$$a_n = 2^{n-1}\left(\frac{2}{\pi}\right)^{1/2}.$$

For $\alpha=\beta=1$, we have *Legendre's polynomials*:

$$\phi_n(x) = \left(\frac{2n+1}{2}\right)^{1/2} P_n(x),$$

$$\phi_n(1) = \left(\frac{2n+1}{2}\right)^{1/2}, \quad \phi_n(-1) = (-1)^n\left(\frac{2n+1}{2}\right)^{1/2}, \tag{27}$$

$$a_n = \frac{1\cdot 3\cdots(2n-1)}{n!}\left(\frac{2n+1}{2}\right)^{1/2}.$$

For $\alpha = \beta = 3/2$,

$$\phi_n(x) = \left(\frac{2}{\pi}\right)^{1/2} \frac{\sin(n+1)\phi}{\sin\phi} \quad (x = \cos\phi)\ ; \quad a_n = 2^n\left(\frac{2}{\pi}\right)^{1/2}. \tag{28}$$

In case of an *infinite interval* we use *polynomials of Laguerre-Tchebycheff*:†

$$(29)\quad \begin{gathered} (a,b) = (0,\infty)\ ;\ p(x) = e^{-hx}\ (h>0), \\ \phi_n(x) = \frac{(-1)^n}{n!} h^{1/2} e^{hx} \frac{d^n}{dx^n}[x^n e^{-hx}] = (-1)^n h^{1/2} \sum_0^n {}_i \binom{n}{i} \frac{(-hx)^i}{i!}\ ; \end{gathered}$$

also the *polynomials of Laplace-Hermite-Tchebycheff*:

$$(a,b) = (-\infty,\infty)\ ;\ p(x) = e^{-hx^2}\ (h>0),$$

$$(30)\quad \begin{aligned} \phi_n(x) &= \left(\frac{h}{\pi}\right)^{1/4} \frac{1}{[(2h)^n\Gamma(n+1)]^{1/2}} e^{hx^2} \frac{d^n}{dx^n}(e^{-hx^2}) \\ &= \left(\frac{h}{\pi}\right)^{1/4} \frac{1}{[(2h)^n\Gamma(n+1)]^{1/2}} \left\{ (2hx)^n - \frac{n(n-1)}{1!}(2hx)^{n-2} \right. \\ &\quad \left. + \frac{n(n-1)(n-2)(n-3)}{2!}(2hx)^{n-4} - \cdots \right\}. \end{aligned}$$

---

* To derive (26), (28) use the equivalent of (4): $\int_{-1}^{1} p(x)\phi_n(x)x^i dx = 0$ ($i = 0, 1, 2, \cdots, n-1$) and substitute $x = \cos\phi$.

† Tchebycheff, *On the development of functions of one variable* (in Russian), Collected Papers, vol. I, pp. 500–508; pp. 504–7.

6. **Asymptotic properties of a certain general class of Tchebycheff polynomials on a finite interval.** We prove the following theorem.

THEOREM III. (i). *With* $p(x)-(S)$ *on* $(-1, 1)$ *we have, for* $n\to\infty$,

$$a_n(p) = 2^n A(p)(1+o(1)), \quad S_n(p) = \sigma(p) + o(1),$$
$$d_{n,n-2} = -(n/4) + d(p) + o(1), d_{n,n-3}(p) = n(d'(p) + o(1)), \cdots *$$

where $A(p)(>0)$, $\sigma(p)$, $d(p)$, $d'(p)$, $\cdots$, *do not depend upon* n.

(ii). *If, on the other hand, there exists a partial interval* (α, β) $(-1\leqq\alpha<\beta<1)$ *such that* $\int_\alpha^\beta p(x)dx=0$, *then* $a_n(p)$ *is of higher order than* 2^n.

The conclusion (i) was established by the writer under more restrictive conditions imposed on $p(x)$.† The proof is based upon the fact that $a_n(p) \backsim 2^n$, which, according to Szegö,‡ holds also for $p(x)-(S)$.

To prove (ii), we note that $a_n(p\,;\,-1,1)\backsim 2^n$ implies that $a_n(p_1;\,0,1)\backsim 4^n$§ ($p_1(x)$ is obtained from $p(x)$ by transforming linearly $(-1, 1)$ into $(0, 1)$), and

$$\int_0^1 \frac{p_1(y)dy}{x-y} = \cfrac{b_1}{x - \cfrac{b_2}{1 - \cfrac{b_3}{x - \cdots}}} \qquad \text{with} \quad b_n \to \tfrac{1}{4} \text{ for } n\to\infty .$$

This shows, according to Blumenthal,‖ that the roots of $\phi_n(p_1\,;\,0,1\,;\,x)$ are *everywhere dense* in $(0, 1)$, which is inconsistent with the above assumption leading to

$$\int_{\alpha_1}^{\beta_1} p_1(x)dx = 0 \qquad (0\leqq\alpha_1<\beta_1\leqq 1),$$

since, in this case, there is at most one root in (α_1, β_1).¶

---

* See (3).

† J. Chokhatte, *Sur le développement de l'intégrale* $\int_a^b \frac{p(y)}{x-y}dy$, Rendiconti del Circolo Matematico di Palermo, vol. 47 (1923), pp. 25–46; pp. 26, 46.

‡ G. Szegö, *Ueber die Entwicklung einer analytischen Funktion*, Mathematische Annalen, vol. 82 (1921), pp. 188–212; pp. 206–7.

§ Chokhatte, Rendiconti del Circolo Matematico di Palermo, loc. cit., pp. 30, 35.

‖ O. Blumenthal, *Ueber die Entwicklung einer willkürlichen Funktion* (Dissertation), Göttingen, 1898, pp. 3–57; pp. 16–17.

¶ Stieltjes, *Recherches sur les quadratures*, Annales de l'Ecole Normale Supérieure, (3), vol. 1 (1884), pp. 409–26; p. 421.

Notes. (1). The expression for $d_{n,n-3}(p)$ not given in the aforesaid paper* is derived in the same way as $d_{n,n-2}(p)$, using Blumenthal's formulas.†

(2). In case of any *finite* (a, b), we get

$$
(31)\qquad \begin{aligned} a_n(p\,;a,b) &= \left(\frac{4}{b-a}\right)^n A(p)[1+o(1)], \\ S_n(p) &= n\left(\frac{b+a}{2}\right) + \sigma(p) + o(1). \end{aligned}
$$

Under certain general conditions the writer has established the following relations:

$$
\left(\frac{2}{\pi(b-a)}\right)^{1/2} A(p_1p_2) = A(p_1)A(p_2)\ ;\ \sigma(p_1p_2) = \sigma(p_1) + \sigma(p_2)\ ;
$$

$$
A(p) = \left(\frac{2}{\pi(b-a)}\right)^{1/2} \exp. -\frac{1}{2\pi}\int_a^b \frac{\log p(x)dx}{[(x-a)(b-x)]^{1/2}},
$$

$$
\sigma(p) = \frac{1}{2\pi}\int_a^b \frac{[x-(a+b)/2]\log p(x)dx}{[(x-a)(b-x)]^{1/2}}.
$$

The formula for $A(p)$ was obtained in an entirely different way by Szegö, loc. cit., p. 207.

The asymptotic behavior of the polynomials under consideration and of their derivatives *outside* $(-1, 1)$ is expressed in the following relations:

$$
(32)\qquad \begin{aligned} \phi_n^{(k)}(z) &= n^k(z+(z^2-1)^{1/2})^{n-k}\mathcal{P}_k(z)(1+o(1)) \\ &\qquad [\,|z|>1\ ;\ z(z^2-1)^{1/2}>0\,], \\ K_n^{(k)}(z) &= n^{2k}[z+(z^2-1)^{1/2}]^{2n-2k}\mathcal{P}(z)[1+o(1)],\ddagger \end{aligned}
$$

where $\mathcal{P}_k(z)$, $\mathcal{P}(z)$ do not depend on n, and $k \geqq 0$, finite. As to points $-1 \leqq z \leqq 1$, we state

THEOREM IV. (i). *Suppose* $-1<z<1$. *Assume there exist finite numbers* $l>-1$, $A>0$, c, d *such that*

$$
(\alpha)\qquad \frac{p(x)}{|x-z|^l} > A \ \textit{for}\ (-1 \leqq)c \leqq x \leqq d(\leqq 1) \qquad (c<z<d).
$$

---

* Chokhatte, Rendiconti del Circolo Matematico di Palermo, loc. cit.

† Blumenthal, loc. cit., p. 11.

‡ J. Chokhatte, *Sur les expressions asymptotiques des polynomes de Tchebycheff et de leurs dérivées*, Comptes Rendus, vol. 183 (1926), pp. 697–99. Cf. also G. Szegö, loc. cit. p., 208.

Let us take, if $l>0$, the smallest l satisfying the above condition. Then $K_n^{(k)}(z) = O(n^{2l'+2k+1})$, *where l' is the smallest integer $\geqq l/2$. In particular,* $K_n^{(k)}(z) = O(n^{2k+1})$ *for* $l \leqq 0$.

(ii). *Suppose* $z^2-1=0$, *say* $z=-1$. *If*

$$(\beta) \qquad \frac{p(x)}{|x+1|^l} > A \text{ for } -1 \leqq x \leqq c(\leqq 1) \qquad (l > -1, A > 0),$$

then $K_n^{(k)}(-1) = O(n^{2l+4k+2})$ (k *finite*, $\geqq 0$ *in* (i), (ii)).

The proof will be omitted, for it is quite similar to that developed for $k=0$ in my paper *On the development of continuous functions*.* The inequalities (22) (see above, p. 575) are made use of.

7. **Order of magnitude of** $K_n^{(k)}(e^{-hx}; 0, \infty; z)$, $K_n^{(k)}(e^{-hx^2}; -\infty, \infty; z)$. Using (4) we obtain readily

$$(33) \quad \begin{aligned} \phi_{2n}(e^{-hx^2}|x|^{2k}; -\infty, \infty; x) &\equiv \phi_n(e^{-hx}x^{k-1/2}; 0, \infty; x^2), \\ \phi_{2n+1}(e^{-hx^2}|x|^{2k}; -\infty, \infty; x) &\equiv x\phi_n(e^{-hx}x^{k+1/2}; 0, \infty; x^2) \end{aligned}$$
$$(h>0;\ k>-\tfrac{1}{2});$$

$$(34) \quad \phi_n^{(k)}(e^{-hx}x^{\beta-1}; 0, \infty; x) = (nh)^{k/2}\phi_{n-k}(e^{-hx}x^{\beta+k-1}; 0, \infty; x)(1+o(1))$$
$$(h, \beta > 0).$$

Formulas (29), (30), (33), (34), combined with the known asymptotic expression (for $n \to \infty$) of $\phi_n(e^{-hx}x^{\beta-1}; 0, \infty; x)$, †give

$$(35) \qquad K_n^{(k)}(e^{-hx}; 0, \infty; z) = O(n^{k+1/2}) \qquad (0 < l \leqq z \leqq L),$$

$$(36) \qquad K_n^{(k)}(e^{-hx^2}; -\infty, \infty; z) = O(n^{k+1/2}) \qquad (0 < l < |z| \leqq L),$$

where l, L are arbitrarily fixed positive quantities. With the material accumulated in §§5–7 we return now to our general formulas (13)–(19).

8. **Polynomials on a finite interval.** We prove the following theorem.

THEOREM V. (i). *For all polynomials $G_n(x)$ satisfying one of the inequalities* (I), (II) *given in the Introduction, with $q(x)$, $p(x)$, $r(x)$ as defined in* (10), (11),

$$|g_n| < \tau 2^{n-1}M, \quad |g_{n-1}| < \tau\cdot 2^{n-1}M, \quad |g_{n-2}| < \tau\cdot 2^{n-2}nM, \cdots,$$
$$|G_n^{(k)}(z)| < \tau n^k |z + (z^2-1)^{1/2}|^{n-k}M$$
$$(|z| > 1,\ z(z^2-1)^{1/2} > 0;\ k \text{ finite}, \geqq 0).$$

---

* These Transactions, vol. 27 (1925), pp. 537–50; p. 540.

† O. Perron, *Ueber das infinitäre Verhalten* · · · , Archiv der Mathematik und Physik, (3), vol. 22 (1914), pp. 329–40; pp. 329–30.

(ii). *If $p(x)$, defined as above, satisfies one of the conditions* (α), (β) *given in Theorem* IV, *then, correspondingly,*

$$(\alpha_1) \qquad |G_n^{(k)}(z)| < \tau n^{l'+k+1/2}M \qquad [(-1 \leqq)c < z < d(\leqq 1)],$$

$$(\beta_1) \qquad |G_n^{(k)}(z)| < \tau n^{l+2k+1}M \qquad (z^2 - 1 = 0).$$

The quantity τ in (i), (ii) *does not depend on n, nor on M.**

Making use of the polynomials of Jacobi (see (25)–(28)), we get from (14)–(19) for any $G_n(x)$:

$$(37)\qquad \begin{aligned} |g_n| &\leqq \left(\int_{-1}^{1}(1+x)^{\alpha-1}(1-x)^{\beta-1}G_n^2(x)dx\right)^{1/2} \cdot 2^{n+(\alpha+\beta)/2-1}\left(\frac{1}{\pi}\right)^{1/2}[1+o(1)],\\ |g_{n-1}| &\leqq \left(\int_{-1}^{1}(1+x)^{\alpha-1}(1-x)^{\beta-1}G_n^2(x)dx\right)^{1/2} \cdot 2^{n+(\alpha+\beta)/2-2}\left(\frac{(\alpha-\beta)^2+1}{\pi}\right)^{1/2}[1+o(1)], \cdots ; \end{aligned}$$

$$G_n^2(z) \leqq \frac{1}{2\pi}\int_{-1}^{1}\frac{G_n^2(x)dx}{(1-x^2)^{1/2}}\left(2n+1+\frac{\xi^{2n+1}-\xi^{-2n-1}}{\xi-\xi^{-1}}\right)$$
$$[|z| \geqq 1;\ \xi = (z^2-1)^{1/2}+z,\ \xi z > 0],$$

$$(38)\qquad \begin{aligned} G_n^2(\pm 1) &\leqq \frac{2n+1}{\pi}\int_{-1}^{1}\frac{G_n^2(x)dx}{(1-x^2)^{1/2}},\\ G_n^2(z) &\leqq \frac{1}{2\pi}\int_{-1}^{1}\frac{G_n^2(x)dx}{(1-x^2)^{1/2}}\left(2n+1+\frac{\sin(2n+1)\phi}{\sin\phi}\right) \end{aligned}$$
$$(|z| \leqq 1;\ z = \cos\phi);$$

$$(39)\qquad [G_n(1)-G_n(-1)]^2 \leqq \frac{4}{\pi}\int_{-1}^{1}\frac{G_n^2(x)dx}{(1-x^2)^{1/2}}\begin{cases} n & (n \text{ even})\\ (n+1) & (n \text{ odd});\end{cases}$$

$$(40)\qquad \begin{aligned} G_n^2(\pm 1) &\leqq \int_{-1}^{1}G_n^2(x)dx\ \frac{(n+1)^2}{2},\\ [G_n(1)-G_n(-1)]^2 &\leqq \int_{-1}^{1}G_n^2(x)dx \begin{cases} n(n+1) & (n \text{ even})\\ (n+1)(n+2) & (n \text{ odd});\end{cases} \end{aligned}$$

the equality holds for certain ploynomials.

* The quantity τ does not depend on z, if, in (i), $-1+\epsilon \leqq z \leqq 1-\epsilon(\epsilon>0$ arbitrarily small, but fixed).

In the more general cases of

$$(41)\quad \int_{-1}^{1} \frac{(1+x)^{\alpha-1}(1-x)^{\beta-1}G_n^2(x)dx}{r(x)} \leqq M^2 \qquad [r(x) - (B) \text{ in } (-1,1)],$$

or

$$(42)\quad \frac{(1+x)^{(\alpha-1)/2}(1-x)^{(\beta-1)/2}\,|\,G_n(x)\,|}{r(x)} < M \text{ for } -1 \leqq x \leqq 1 \qquad [r(x) - (I) \text{ in } (-1,1)],$$

the right-hand members of (37)–(40) acquire the factor R given in (V). It is interesting to note that (i) in Theorem V gives for $|g_k|$ and $G_n(z)$ expressions of the same order with respect to n as those given in the particular case $q(x) \equiv 1$, by Tchebycheff* and W. Markoff.†

We close the case of a finite interval with one more application of our general formulas. The relation (18) gives for $(a, b) = (-1, 1)$, $k = n$, $G_n(x) = P_n(x)$ (Legendre's polynomial):

$$g_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} < 2^{n+1}\left(\frac{1}{\pi(2n+1)}\right)^{1/2} \quad \left(\text{since } \int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n+1}\right).$$

On the other hand, (23) gives with $p(x) \equiv 1$,

$$\frac{1 \cdot 3 \cdots (2n-1)}{n!}\left(\frac{2n+1}{2}\right)^{1/2} > 2^{n-3/2}.$$

Thus we get interesting inequalities:

$$2^{n-1/2}\left(\frac{1}{2n+1}\right)^{1/2} < \frac{1 \cdot 3 \cdots (2n-1)}{n!} < 2^{n+1}\left(\frac{1}{\pi(2n+1)}\right)^{1/2}.$$

9. **Polynomials in an infinite interval.** We prove the following theorem.

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* Tchebycheff, *On functions deviating the least from zero*, Collected Papers, vol. II, No. 18, pp. 335–56; p. 343.

† W. Markoff, *Ueber Polynome, die in einem gegebenem Intervalle möglichst wenig von Null abweichen*, Mathematische Annalen, vol. 77 (1916), pp. 213–58; p. 248.

Theorem VI. (i). *For all polynomials $G_n(x)$ satisfying the inequality*

$$\int_0^\infty e^{-hx}G_n^2(x)dx \leqq M^2 \qquad (h > 0, \text{ equality attained}),$$

$$|g_n| \leqq \frac{h^{n+1/2}}{n!}M, \quad |g_{n-1}| \leqq \frac{h^{n-1/2}}{(n-1)!}\left[\binom{n}{n-1}^2 + \binom{n-1}{n-1}^2\right]^{1/2}M,$$

$$|g_{n-2}| \leqq \frac{h^{n-3/2}}{(n-2)!}\left[\binom{n}{n-2}^2 + \binom{n-1}{n-2}^2 + \binom{n-2}{n-2}^2\right]^{1/2}M, \cdots \qquad (\text{see } (29));$$

the equality holds for certain polynomials.

(ii). *In cases of one of the inequalities* (I), (II) *given in the Introduction with* $q(x) = e^{-hx}/r(x)$ $(r(x)-(B)$ *in* $(0, \infty))$ *or with*

$$q^2(x) = e^{-hx}/r(x) \qquad (r(x) - (I) \text{ in } (0, \infty)),$$

we have

$$|g_k| \leqq R\frac{h^{k+1/2}}{k!}\binom{n}{k}M(1 + o(1)) \qquad (n - k \text{ finite})$$

$$|g_k| \leqq R\frac{h^{k+1/2}}{k}\binom{n}{k}(n - k + 1)^{1/2}M \qquad (k \geqq 0, \text{ arbitrary})$$

where

$$R^2 = r_{\max}, \quad \int_0^\infty r(x)dx \quad \text{respectively};$$

$$[G_n^{(k)}(z)]^2 < \tau M^2 n^{k+1} \qquad (k \text{ finite}, \geqq 0;\ 0 < l \leqq z \leqq L).$$

(iii). *Replace* $(0, \infty)$, e^{-hx} *by* $(-\infty, \infty)$, e^{-hx^2} *respectively. Then*

$$|g_n| \leqq R\left(\frac{h}{\pi}\right)^{1/4}\left(\frac{(2h)^n}{n!}\right)^{1/2}M, \qquad |g_{n-1}| \leqq R\left(\frac{h}{\pi}\right)^{1/4}\left(\frac{(2h)^{n-1}}{(n-1)!}\right)^{1/2}M,$$

$$|g_{n-2}| \leqq R\left(\frac{h}{\pi}\right)^{1/4}\left(\frac{(2h)^{n-2}}{(n-2)!}\right)^{1/2}\left(\frac{n(n-1)}{(2h)^2} + 1\right)^{1/2}M, \cdots \qquad (\text{see } (30)),$$

where $R^2 = 1$ *in* (i), $r_{\max}$, $\int_{-\infty}^{\infty} r(x)dx$ *respectively in* (ii);

$$[G_n^{(k)}(z)]^2 < \tau M^2 n^{k+1/2} \qquad (0 < l \leqq |z| \leqq L).$$

Here l, L are arbitrarily fixed, τ does not depend on n, nor on M, nor on z.

10. **Inequalities for $a_n(p)$ on an infinite interval.** We give, in closing, upper and lower limits for $a_n(p)$ on an infinite interval. We get from (20), taking respectively (see (29), (30),

$$(a,b) = (0,\infty),\ q(x) = e^{-hx}x^{\beta-1}(h,\beta > 0) \text{ with } a_n(q) = \frac{h^{n+\beta/2}}{(\Gamma(n+1)\Gamma(n+\beta))^{1/2}},$$

$$(a,b) = (-\infty,\infty),\ q(x) = e^{-hx^2},$$

$$(43)\qquad \frac{h^{n+\beta/2}}{(\Gamma(n+1)\Gamma(n+\beta))^{1/2}}\frac{1}{(r_{\min})^{1/2}} > a_n(e^{-hx}x^{\beta-1}r(x)\ ;\ 0,\ \infty)$$

$$> \frac{h^{n+\beta/2}}{(\Gamma(n+1)\Gamma(n+\beta))^{1/2}}\frac{1}{(r_{\max})^{1/2}},$$

$$a_n(e^{-hx}x^{\beta-1}r(x)\ ;\ 0,\infty) > \frac{h^{n+\beta/2}}{(\Gamma(n+1)\Gamma(n+\beta))^{1/2}\left(\int_0^\infty r(x)dx\right)^{1/2}};$$

$$(44)\qquad \left(\frac{h}{\pi}\right)^{1/4}\left(\frac{(2h)^n}{n!}\right)^{1/2}\frac{1}{(r_{\min})^{1/2}} > a_n(e^{-hx^2}r(x)\ ;-\infty,\infty)$$

$$> \left(\frac{h}{\pi}\right)^{1/4}\left(\frac{(2h)^n}{n!}\right)^{1/2}\frac{1}{(r_{\max})^{1/2}},$$

$$a_n(e^{-hx^2}r(x)\ ;-\infty,\infty) > \left(\frac{h}{\pi}\right)^{1/4}\left(\frac{(2h)^n}{n!}\right)^{1/2}\frac{1}{\left[\int_{-\infty}^\infty r(x)dx\right]^{1/2}}.$$

11. We could multiply our results, utilizing many other classes of Tchebycheff polynomials (e.g., $\phi_n[(1+x)^{\alpha-1}(1-x)^{\beta-1}|x|^{2k};\ -1,\ 1\ ;x]$, $\phi_n(e^{-hx}x^{k-1};\ 0,\ \infty\ ;\ x)$ $(k>0)$, etc.). However, the above results suffice to show that any advance in the theory of Tchebycheff polynomials leads to new results in the general theory of polynomials, and conversely. Moreover, we see easily that our results hold for polynomials with *complex coefficients* (x being real), mutatis mutandis.

I have shown elsewhere the extension of the method herein employed to the case of an arbitrary polynomial of degree $\leqq n$ *in several variables*, $\omega(G_n)$ *being an arbitrarily given homogeneous expression of any degree whatsoever of its coefficients.**

* J. Chokhatte, *Sur quelques applications des polynomes de Tchebycheff à plusieurs variables*, Comptes Rendus, vol. 183 (1926), pp. 442–44.

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A FACTORIZATION THEORY FOR FUNCTIONS $\sum_{i=1}^{n} a_i e^{\alpha_i x}$*

BY

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Introduction. When, in the expression

$$a_0 e^{\alpha_0 x} + \cdots + a_n e^{\alpha_n x},$$

we allow n to assume all positive integral values, and the a's and α's all constant values, we obtain a class of functions which is closed with respect to multiplication; that is, the product of any two functions of the class is also in the class. There arises thus the problem of determining all possible representations of a given function of the class as a product of functions of the class. This problem is solved in the present paper.

To secure a simple statement of results, we subject our functions to some adjustments. Let the terms in each function be so arranged that α_i comes before α_j if the real part of α_i is less than that of α_j, or if the real parts are equal but the coefficient of $(-1)^{1/2}$ in α_i is less than that in α_j.† With this arrangement, it is evident that the first term in a product of several functions is the product of the first terms of those functions. Thus we do not specialize our problem if we limit ourselves to functions with first term unity ($a_0=1$, $\alpha_0=0$), resolving such functions into factors‡ with first term unity. We shall make this limitation, and shall furthermore admit into our work only functions with more than one term, that is, functions distinct from unity.§

Our first theorem states that *if*

$$1 + a_1 e^{\alpha_1 x} + \cdots + a_n e^{\alpha_n x} \tag{1}$$

is divisible by

$$1 + b_1 e^{\beta_1 x} + \cdots + b_r e^{\beta_r x},$$

with no b equal to zero, then every β is a linear combination of $\alpha_1, \cdots, \alpha_n$ with rational coefficients.

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† We assume, of course, that the α's in a function are distinct from one another.

‡ If $P=P_1 \cdots P_m$, with each P_i a function of our class, P will be said to be *divisible* by each P_i, and each P_i will be called a *factor* of P.

§ One might ask whether unity has factors (of first term unity) which are distinct from unity. That it has none follows from the fact that the last term of the product of several functions is the product of the last terms of those functions.

We shall say that the function (1) is *simple* if there exists a number λ of which every α is an integral multiple. It is easy to see that every simple function has an infinite number of factors. In short, as no α has a negative real part, we may suppose λ to be such that every α is a *positive* integral multiple of λ. For every positive integer r, the simple function is a polynomial in $e^{\lambda x/r}$ of degree at least r. It therefore has at least r factors of the form $1+ce^{\lambda x/r}$. It is a consequence of the theorem stated above that every factorization of a simple function is found in this way.

There exist, in abundance, functions (1) which are not divisible by functions (1) other than themselves. We shall call such functions *irreducible.*

We may now state our theorem of factorization.

THEOREM. *Every function*

$$1+a_1e^{\alpha_1 x}+\cdots+a_ne^{\alpha_n x},$$

distinct from unity, can be expressed in one and in only one way as a product

$$(S_1S_2\cdots S_s)(I_1I_2\cdots I_t),$$

in which $S_1,\cdots,S_s$ are simple functions such that the coefficients of x in any one of them have irrational ratios to the coefficients of x in any other, and in which $I_1,\cdots,I_t$ are irreducible functions.*

Most of our work centers about the proof that a resolution exists. Because a function may have an infinite number of factors, this resolution cannot be accomplished by the process of repeated factorization used in the proofs of most factorization theorems. The uniqueness is easy to establish.

1. **Exponents of factors.** We understand, in everything which follows, that the terms in our functions are ordered in the manner explained in the introduction. Of course, the real part of every coefficient of x will be greater than or equal to zero, and, when the real part is zero, the coefficient of $(-1)^{1/2}$ will be positive.

THEOREM. *If $1+\sum_{i=1}^{n} a_ie^{\alpha_i x}$ is divisible by $1+\sum_{i=1}^{r} b_ie^{\beta_i x}$, with no b zero, then every β is a linear combination of $\alpha_1,\cdots,\alpha_n$ with rational coefficients.*

Let

$$\begin{aligned}1+a_1e^{\alpha_1 x}&+\cdots+a_ne^{\alpha_n x}\\ &=(1+b_1e^{\beta_1 x}+\cdots+b_re^{\beta_r x})(1+c_1e^{\gamma_1 x}+\cdots+c_se^{\gamma_s x}),\end{aligned}\tag{2}$$

with no b or c equal to zero. Suppose that there exists a β, say β_j, which is not linear in the α's with rational coefficients.

* That is, the α's.

We shall call a set of numbers $m_1, \cdots, m_p$ *independent* if there does not exist a relation $\sum q_i m_i = 0$, with the q's rational, and not all zero.

Let $m_1, \cdots, m_p$ be an independent set of numbers such that every α is a linear combination of the m's with rational coefficients. We shall use the symbol m_0 to represent the β_j considered above. Then $m_0, m_1, \cdots, m_p$ are independent.

We can certainly adjoin new m's to those we already have so as to form an independent set

$$m_0, m_1, \cdots, m_p, \cdots, m_t$$

such that every α, β and γ is linear in the numbers of this set with rational coefficients.

Every β has a *unique* representation of the form $\sum_{i=0}^{t} q_i m_i$, with rational q's. We select those β's for which q_0 is a maximum, say u_0, of those selected we pick out such for which q_1 is a maximum, say u_1, and continue in this fashion for all the q's, obtaining a certain β, call it B, with a representation $\sum u_i m_i$. Because $\beta_j = m_0$, we have $u_0 \geqq 1$.

We now adjoin to the γ's a $\gamma_0 = 0$, and call the term unity, in the second factor of the right member of (2), $e^{\gamma_0 z}$. Of course, γ_0 is linear in the m's with zero coefficients. Similarly, we adjoin a β_0 to the β's.

We choose from among the γ's a $C = \sum v_i m_i$ with the v's determined successively as maxima. Because $\gamma_0 = 0$, we have $v_0 \geqq 0$.

The multiplication of the factors in the second member of (2) yields a term in $e^{(B+C)z}$. From the manner in which B and C are determined, we see that $B+C$ cannot equal any other $\beta+\gamma$. Hence the term in $e^{(B+C)z}$ does not cancel out and $B+C$ must be an α.

But as the expression for $B+C$ in the m's involves m_0 with a coefficient at least unity, and as the α's depend only on $m_1, \cdots, m_p$, the equality of $B+C$ with an α would imply that the m's are not independent. This proves that the β's are linear in the α's with rational coefficients.

2. **Selection of basis.** We are going to prove the existence of an independent set of numbers $\mu_1, \cdots, \mu_p$, such that every α is a linear combination of the μ's with *positive* rational coefficients.

Each α has either a positive real part, or a zero real part and a positive coefficient for $(-1)^{1/2}$. Thus, if δ is a sufficiently small positive quantity, the product of each α by $e^{-\delta(-1)^{1/2}}$ will have a positive real part. We choose such a δ, and let $A_i = e^{-\delta(-1)^{1/2}}\alpha_i$ $(i = 1, \cdots, n)$.

Let $m_1, \cdots, m_p$ be any independent set of numbers in terms of which the A's can be expressed linearly with rational coefficients. Suppose that

$$A_i = q_{i1} m_1 + \cdots + q_{ip} m_p \qquad (i = 1, \cdots, n), \tag{3}$$

all q's being rational. We shall determine an independent set of numbers $M_1, \cdots, M_p$ such that the m's are linear in the M's with rational coefficients, and such that the coefficients in the expressions of the A's, in terms of the M's, found from (3), are all positive.

We associate with each m_i $(i=1, \cdots, p)$, p rational numbers t_{ij} $(j=1, \cdots, p)$, choosing for each t_{ij} a rational number close to the real part of m_i (how close the approximation should be will be made clear below), and taking care that the determinant $|t_{ij}|$ does not vanish. We determine M's through the equations

$$(4) \qquad m_i = t_{i1}M_1 + \cdots + t_{ip}M_p \qquad (i = 1, \cdots, p).$$

The coefficient of M_j in the expression for A_i in terms of the M's is

$$q_{i1}t_{1j} + q_{i2}t_{2j} + \cdots + q_{ip}t_{pj}.$$

If t_{ij} is very close to the real part of m_i, this coefficient will be, according to (3), very close to the real part of A_i, and will therefore be positive.

The M's are independent. For, let a relation

$$(5) \qquad Q_1M_1 + \cdots + Q_pM_p = 0$$

hold, with the Q's rational and not all 0. Because $|t_{ij}| \neq 0$, the equations

$$t_{1j}q_1 + \cdots + t_{pj}q_p = Q_j \qquad (j = 1, \cdots, p)$$

give a set of rational values, not all 0, for $q_1, \cdots, q_p$. Hence (5) implies an impossible relation $\sum q_i m_i = 0$.

If now we put $\mu_i = e^{\delta(-1)^{1/2}} M_i$ $(i=1, \cdots, p)$, we have an independent set of quantities $\mu_1, \cdots, \mu_p$ of which every $\alpha_i = e^{\delta(-1)^{1/2}} A_i$ is a linear combination with *positive* rational coefficients.

In what follows, we shall use only the fact that the coefficients just secured are *non-negative.*

3. **Expressions for β's and γ's.** Of course, every γ, as well as every β, in (2), is linear in the α's with rational coefficients. We say that, in the expression for each β and γ in terms of the μ's found in § 2, the coefficients are all non-negative. Let the contrary be assumed, and to fix our ideas, suppose that some β involves a μ with a negative coefficient. As we have perfect freedom in assigning subscripts to the μ's, we assume that some β involves μ_1 with a negative coefficient.

Of all β's, we select those for which the coefficient of μ_1 is a minimum, of those selected we take such for which the coefficient of μ_2 is a minimum, and continue in this fashion until μ_p is determined as a minimum. We find in this way a definite β, call it B, *with a negative coefficient for* μ_1.

We now adjoin to the γ's a $\gamma_0=0$, and regard the term unity in the second factor of the second member of (2) as being $e^{\gamma_0 x}$. We find, as above, a γ, call it C, with coefficients determined successively as minima. The coefficient of μ_1 in C is not positive, for $\gamma_0=0$.

On multiplying together the factors in the second member of (2), we find a term in $e^{(B+C)x}$, which cannot be cancelled. Hence $B+C$ must be an α. This is impossible, because the coefficient of μ_1 in $B+C$ is negative. Our statement is proved.

More generally, let the first term of (2) be represented by P, and suppose that

$$P = P_1 \cdots P_m, \tag{6}$$

where each subscripted P is, like P itself, a function of the form (1). When μ's are chosen as in § 2, each coefficient of x in each P_i is linear in the μ's with non-negative rational coefficients.

4. **The identities.** As we may replace the μ's by any submultiples of themselves, we may assume that the coefficients of x in the first member of (6) are linear in the μ's with non-negative *integral* coefficients. We make this assumption.

We now associate with each $e^{\mu_i x}$ a variable y_i. We express each exponential in (6) as a product of non-negative rational powers of the exponentials $e^{\mu_i x}$, and replace each $e^{\mu_i x}$ by y_i.

Equation (6) becomes a relation in the y's which holds when each y_i is replaced by $e^{\mu_i x}$. We say that this relation in the y's is an identity in the y's.*

If it were not, there would exist a sum of rational powers of the y's, not identically zero, which would vanish when each y_i is replaced by $e^{\mu_i x}$. But, because the μ's are independent, any two of the products of powers of the y's would yield terms of the form he^{kx} (h and k constants), with distinct k's. As a sum

$$h_1 e^{k_1 x} + \cdots + h_q e^{k_q x}$$

cannot vanish for every x if the k's are distinct from one another and the h's are not all zero, our statement that the relation in the y's is an identity is proved.

5. **The polynomial problem.** We may replace each y_i by a positive integral power of itself in such a way that the sums of rational powers of

* If more than one fractional power of a y_i appears in the relation, the exponents should be reduced to a common denominator, and the various powers of the y_i regarded as integral powers of a single fractional power of the y_i.

the y's obtained from the P_i's of (6) go over into polynomials in the y's. The relation in the y's thus found is, of course, an identity.

We have now a method for obtaining every representation of P as a product $P_1 \cdots P_m$. First we find an independent set of μ's in terms of which the coefficients of x in P can be expressed linearly, with non-negative integral coefficients. We then replace each $e^{\mu_i x}$ in P by a variable y_i, so that P becomes associated with a polynomial $Q(y_1, \cdots, y_p)$. We replace the y's, in all possible ways, by positive integral powers of themselves, obtaining a family of polynomials $Q(y_1^{t_1}, \cdots, y_p^{t_p})$. To each resolution of each of the latter polynomials into factors with first term unity, there corresponds a factorization of P.* All factorizations of P are found in this way.

In our study of $Q(y_1, \cdots, y_p)$ and of the polynomials derived from it, we may limit ourselves to the case in which Q is irreducible. For, if Q is reducible, the factorizations of every polynomial obtained from it by replacing the y's by powers of themselves can be obtained by resolving Q into its irreducible factors, replacing the y's by powers of themselves in those factors, and factoring the polynomials thus obtained.

Our problem thus becomes: *Given an irreducible polynomial $Q(y_1, \cdots, y_p)$, to determine for which positive integers $t_1, \cdots, t_p$ the polynomial $Q(y_1^{t_1}, \cdots, y_p^{t_p})$ is reducible.*

6. **Primary polynomials.** Let $Q(y_1, \cdots, y_p)$ be a polynomial in $y_1, \cdots, y_p$, more definitely, a sum of products of non-negative integral powers of $y_1, \cdots, y_p$, with constant coefficients *distinct from zero*. It is understood that each y_i figures in some term with an exponent greater than zero.

If the highest common factor of all the exponents of y_i in Q is unity, we shall say that Q is *primary* in y_i. If Q is primary in each of its variables, we shall say, simply, that Q is *primary*.

There exists one and only one set of positive integers $t_1, \cdots, t_p$ such that Q can be written in the form $Q'(y_1^{t_1}, \cdots, y_p^{t_p})$, with $Q'(y_1, \cdots, y_p)$ primary. In short, t_i can and must be taken as the highest common factor of the exponents of y_i in Q.

Let $Q(y_1, \cdots, y_p)$ be an irreducible polynomial whose first term is unity. Let $t_1, \cdots, t_p$ be any positive integers. It is evident that every factor of

* The question arises as to whether the coefficients of x obtained, when each y_i is replaced in the factors of $Q(y_1^{t_1}, \cdots, y_p^{t_p})$ by $e^{\mu_i x}$, have positive real parts or zero real parts and positive coefficients for $(-1)^{1/2}$. That the answer is affirmative follows from the facts that unity is a term of each function obtained, and that the first term of a product is the product of the first terms.

$Q(y_1^{t_1}, \cdots, y_p^{t_p})$ has a term independent of the y's. Suppose then that

$$Q(y_1^{t_1}, \cdots, y_p^{t_p}) = Q_1Q_2 \cdots Q_m$$

with each Q_i an irreducible polynomial* in $y_1, \cdots, y_p$ with first term unity.

We associate with each i $(i=1, \cdots, p)$, a primitive t_ith root of unity, ϵ_i. The polynomial $Q(y_1^{t_1}, \cdots, y_p^{t_p})$ undergoes no change when each y_i is replaced by $\epsilon_i^{a_i}y_i$, the a's being any integers. Hence, for such a substitution, the Q_i's go over into constant factors times one another. As each Q_i has unity for a term, the constant factors are unity, so that the Q_i's are interchanged among themselves.

We say that, given any Q_i, there is a substitution of the type described above which converts Q_1 into Q_i. For, suppose that Q_1 is converted only into $j<m$ of the functions, say $Q_1, \cdots, Q_j$. Then the substitutions interchange $Q_1, \cdots, Q_j$ among themselves. Hence the product $Q_1 \cdots Q_j$ is invariant under all of the substitutions.† This means that $Q_1 \cdots Q_j$ is a rational integral function of $y_1^{t_1}, \cdots, y_p^{t_p}$, and hence that $Q(y_1, \cdots, y_p)$ is reducible. Thus Q_1 goes over into every Q_i.

Hence, if Q_1 is primary in certain variables, every Q_i will be primary in those variables.

Similarly, if Q is primary in certain variables, every Q_i will be primary in those variables.

7. **The first lemma.** LEMMA. *Let $Q(y_1, \cdots, y_p)$ be a primary, irreducible polynomial, of degree δ, consisting of more than two terms and with unity for its term of lowest degree. Suppose that, for certain positive integers $t_1, \cdots, t_p$, the irreducible factors of $Q(y_1^{t_1}, \cdots, y_p^{t_p})$ are primary. Then there exist a polynomial $T(y_1, \cdots, y_p)$ and positive integers $\tau_1, \cdots, \tau_p$ which have the following properties*:

(a) *$T(y_1, \cdots, y_p)$ is primary and irreducible, with unity for its term of lowest degree.*

(b) *The degree of $T(y_1, \cdots, y_p)$, in each variable, does not exceed the corresponding degree of $Q(y_1, \cdots, y_p)$.*

(c) *For every i, $\tau_i/t_i \geqq \delta^{-p}$.*

(d) *The irreducible factors of $T(y_1^{\tau_1}, \cdots, y_p^{\tau_p})$ are primary and consist of more than two terms.*

(e) *The polynomials $T(y_1, y_2^{\tau_2}, \cdots, y_p^{\tau_p})$, $T(y_1^{\tau_1}, y_2, y_3^{\tau_3}, \cdots, y_p^{\tau_p})$, $\cdots$, $T(y_1^{\tau_1}, y_2^{\tau_2}, \cdots, y_{p-1}^{\tau_{p-1}}, y_p)$ are all irreducible.*

* The term "polynomial" is being used here in its usual sense, rather than in the sense explained at the head of this section. It will be seen, however, that each Q_i involves every y, so that each Q_i is also a polynomial in $y_1, \cdots, y_p$ in the stricter sense.

† This is true even when $Q_1, \cdots, Q_j$ are not distinct.

For simplicity of notation, we shall take the case of $p=3$; it will be seen that the proof is general.

We write x, y, z instead of y_1, y_2, y_3, and p, q, r instead of t_1, t_2, t_3. We shall show the existence of a $T(x, y, z)$ and of integers π, χ, ρ which have the qualities claimed for T, τ_1, etc. in the statement of our lemma.

Let

$$(7) \qquad Q(x, y^q, z^r) = Q_1 \cdots Q_m$$

with each Q_i an irreducible polynomial having unity for a term.

Every Q_i is obtained from Q_1 by replacing y by y times a qth root of unity and z by z times an rth root of unity.

Certainly Q_1 is primary in x. It may or may not be primary in y and in z. Let

$$Q_1 = R(x, y^{q_1}, z^{r_1}),$$

with $R(x, y, z)$ primary. Certainly $R(x, y, z)$ is irreducible.

Let the degree of $Q(x, y, z)$ in x be a. We say that $q/q_1 \leqq a$ and $r/r_1 \leqq a$. First m, the number of Q_i's, does not exceed a, because every Q_i contains x. Certainly q is divisible by q_1. Let $k=q/q_1$, and let ϵ be a primitive kth root of unity. Because $R(x, y, z)$ is primary, the k polynomials $R(x, \epsilon^i y^{q_1}, z^{r_1})$, $i=1, \cdots, k$, are all distinct. But as each ϵ^i is a q_1th power of a qth root of unity, each of these polynomials is some Q_i. Hence $k \leqq m$, so that $q/q_1 \leqq a$. Similarly, $r/r_1 \leqq a$.

Let b and c be the respective degrees of $Q(x, y, z)$ in y and z, and a_1, b_1, c_1 the respective degrees of $R(x, y, z)$ in x, y, z. We have, by (7), $a=ma_1$, so that $a_1 \leqq a$. Now, as $mb_1q_1=bq$, and as $q \leqq mq_1$ (proved above), we have $b_1 \leqq b$. Similarly, $c_1 \leqq c$.

Let p_1 be written instead of p. Consider the polynomial $R(x^{p_1}, y, z^{r_1})$. Let

$$R(x^{p_1}, y, z^{r_1}) = R_1 \cdots R_{m'},$$

with each R_i an irreducible polynomial having unity for a term.

Certainly R_1 is primary in y. It may not be primary in x and in z. Let

$$R_1 = S(x^{p_2}, y, z^{r_2})$$

with $S(x, y, z)$ primary. Of course, $S(x, y, z)$ is irreducible. We show as above that $p_1/p_2 \leqq b_1$, $r_1/r_2 \leqq b_1$, and that a_2, b_2, c_2, the degrees of $S(x, y, z)$ in x, y, z, are respectively not greater than a_1, b_1, c_1.

Let q_2 be written in place of q_1. We are going to prove that $S(x, y^{q_2}, z^{r_2})$ is irreducible.

We recall that $Q_1=R(x, y^{q_1}, z^{r_1})$ is irreducible. Suppose that $S(x, y^{q_2}, z^{r_2})$ is reducible. Then $R_1(x, y^{q_1}, z)$, which equals $S(x^{p_2}, y^{q_2}, z^{r_2})$, can be factored into the form

$$A(x^{p_2}, y, z)B(x^{p_2}, y, z),$$

with $A(x, y, z)$ and $B(x, y, z)$ non-constant rational integral functions.

Let $k=p_1/p_2$ and let ϵ be a primitive kth root of unity. Because $S(x, y, z)$ is primary, the k polynomials $S(\epsilon^i x^{p_2}, y^{q_2}, z^{r_2})$, $i=1, \cdots, k$, are distinct. But as each ϵ^i is a p_2th power of a p_1th root of unity, each of the k polynomials is obtained from $R_1(x, y^{q_1}, z)$ by replacing x by x times a p_1th root of unity. Hence each of the polynomials is of the form $R_i(x, y^{q_1}, z)$.

Thus, the product of the k functions $A(\epsilon^i x^{p_2}, y, z)$, $i=1, \cdots, k$, is a factor of $R(x^{p_1}, y^{q_1}, z^{r_1})$, which function equals $Q_1(x^{p_1}, y, z)$. But the product is rational in x^{p_1}, y and z. Thus $Q_1(x, y, z)$ must be reducible. This proves that $S(x, y^{q_2}, z^{r_2})$ is irreducible.

Now, let

$$S(x^{p_2}, y^{q_2}, z) = S_1 \cdots S_{m''},$$

with each S_i an irreducible polynomial having unity for a term. Let

$$S_1 = T(x^{\pi}, y^{\chi}, z),$$

with $T(x, y, z)$ primary (and irreducible). We prove as above that the degree of $T(x, y, z)$ in each variable is not greater than the corresponding degree of $S(x, y, z)$, and that $p_2/\pi \leqq c_2$, $q_2/\chi \leqq c_2$.

Let ρ stand for r_2. It can be shown, as above, that $T(x, y^{\chi}, z^{\rho})$ and $T(x^{\pi}, y, z^{\rho})$ are irreducible (Item (e)).

We wish to show that the irreducible factors of $T(x^{\pi}, y^{\chi}, z^{\rho})$ are primary. That function is a factor of $S(x^{p_2}, y^{q_2}, z^{r_2})$ which is a factor of $R(x^{p_1}, y^{q_1}, z^{r_1})$, a factor of $Q(x^p, y^q, z^r)$. As the irreducible factors of the latter function are primary, those of $T(x^{\pi}, y^{\chi}, z^{\rho})$ are also.

We shall show that each irreducible factor of $T(x^{\pi}, y^{\chi}, z^{\rho})$ contains more than two terms. Let

$$T(x^{\pi}, y^{\chi}, z^{\rho}) = T_1 \cdots T_t,$$

each T_i being irreducible, with unity for its term of lowest degree. Suppose that T_1 has just two terms, and let

$$T_1 = 1 + cx^{\alpha}y^{\beta}z^{\gamma}.$$

Because T_1 is an irreducible factor of $Q(x^p, y^q, z^r)$, the other irreducible factors of $Q(x^p, y^q, z^r)$ are found by multiplying the variables in T_1 by roots of unity. Hence $Q(x^p, y^q, z^r)$ is a polynomial in the product $x^{\alpha}y^{\beta}z^{\gamma}$. Thus the

exponents of x, y and z in each term of $Q(x, y, z)$ are respectively proportional to α/p, β/q, γ/r.

Let A be the highest common factor of all the exponents of x which appear in $Q(x, y, z)$, and let B and C be the highest common factors for y and z respectively. Then A, B, C are proportional to $\alpha/p, \beta/q, \gamma/r$, so that $Q(x, y, z)$ is a polynomial in the product $x^A y^B z^C$. Then $Q(x, y, z)$, which has more than two terms, is reducible, for any polynomial in one variable, of more than two terms, is reducible. This absurdity shows that T_1 has more than two terms.

The ratios π/p, χ/q, ρ/r are each at least equal to $1/ab_1c_2 \geqq 1/abc$, and hence are at least equal to δ^{-3}.

The proof of the lemma is completed.

8. **The second lemma.** Lemma. *Let $Q(y_1, \cdots, y_p)$ be a primary irreducible polynomial, consisting of more than two terms, and having unity for its term of lowest degree. There exist only a finite number of sets of positive integers $t_1, \cdots, t_p$ such that the irreducible factors of $Q(y_1^{t_1}, \cdots, y_p^{t_p})$ are primary.*

We use the polynomial T and the integers $\tau_1, \ldots, \tau_p$ whose existence was shown in § 7. Let

$$T(y_1^{\tau_1}, \cdots, y_p^{\tau_p}) = T_1 \cdots T_t, \tag{8}$$

with each T_i a primary irreducible polynomial, of more than two terms, with unity for its first term.

Our first step will be to prove that

$$t = \tau_1 = \tau_2 = \cdots = \tau_p,$$

t being the number of factors in the second member of (8). Let ϵ be a primitive τ_1th root of unity. Then the τ_1 polynomials $T_1(\epsilon^j y_1, y_2, \cdots, y_p)$, $j=1, \cdots, \tau_1$, are all distinct, and are among the polynomials T_i. The product of these polynomials is a polynomial in $y_1^{\tau_1}, y_2, \cdots, y_p$ which is a factor of the first member of (8). Hence, if τ_1 were less than t, $T(y_1, y_2^{\tau_2}, \cdots, y_p^{\tau_p})$ would be reducible. Thus $\tau_1 = t$. Similarly, $\tau_2 = t$, etc.

It cannot be that there exist numbers $\lambda_1, \cdots, \lambda_p$ such that, in every term of T_1, the exponents of $y_1, \cdots, y_p$ are respectively proportional to $\lambda_1, \cdots, \lambda_p$. As was shown in § 7, the existence of such λ's would imply the reducibility of T_1.

Let us suppose, then, fixing our ideas, that

$$Ay_1^{\alpha_1}y_2^{\alpha_2} \cdots y_p^{\alpha_p}, \qquad By_1^{\beta_1}y_2^{\beta_2} \cdots y_p^{\beta_p}$$

(A and B constants), are two terms of T_1 with α_1 and α_2 not proportional to β_1 and β_2, that is, with $\alpha_1\beta_2-\beta_1\alpha_2\neq 0$.

As we are free to interchange the letters α and β, we assume that $\alpha_1\beta_2-\beta_1\alpha_2>0$. Then $\beta_2>0$.

There are t^2 ways of multiplying y_1 and y_2 in T_1 by tth roots of unity. As this group of t^2 operations converts T_1 into precisely t distinct polynomials, there must be t of the operations which leave T_1 invariant.

Let

$$y_1' = \epsilon^u y_1, \qquad y_2' = \epsilon^v y_2$$

be any of the t operations which leave T_1 invariant. Then the pair of congruences

$$\alpha_1 u + \alpha_2 v \equiv 0,$$
$$\beta_1 u + \beta_2 v \equiv 0 \qquad (\text{mod } t),$$

must have at least t solutions in common, u and v being, in each solution, non-negative integers less than t.

Any solution of the above congruences is also a solution of the congruences

$$(\alpha_1\beta_2 - \beta_1\alpha_2)u \equiv 0, \tag{9}$$
$$v\beta_2 \equiv -\beta_1 u \qquad (\text{mod } t). \tag{10}$$

Let h be the highest common factor of $\alpha_1\beta_2-\beta_1\alpha_2$ and t. Then (9) has precisely h solutions in u. Let k be the highest common factor of β_2 and t. Then, for each u satisfying (9), the congruence (10) has at most k solutions in v.*

Hence

$$hk \geqq t,$$

so that either $h\geqq t^{1/2}$ or $k\geqq t^{1/2}$.

Suppose first that $h\geqq t^{1/2}$. Then $\alpha_1\beta_2-\beta_1\alpha_2$ is at least $t^{1/2}$, so that either α_1 or β_2 is at least $t^{1/4}$.

Suppose that $\alpha_1\geqq t^{1/4}$. Then the degree of T_1 is at least $t^{1/4}$. Let a be the degree of $T(y_1, \cdots, y_p)$ in y_1. Then, by (8),

$$at \geqq t\cdot t^{1/4}.$$

We know that a does not exceed the degree of Q in y_1. Hence $a\leqq\delta$, where δ is the degree of Q. Then $t\leqq\delta^4$, so that, by the lemma of § 7, $t_1, \cdots, t_p$ are each not greater than δ^{p+4}.

We find the same bound for t_1 etc. when $\beta_2\geqq t^{1/4}$.

If $k\geqq t^{1/2}$, then β_2 must be at least $t^{1/2}\geqq t^{1/4}$.

* Accurately, either no solutions or k solutions.

We have thus shown that none of the exponents $t_1, \cdots, t_p$ can exceed δ^{p+4}. This proves our lemma.

9. **The factorization theorem.** We proceed now to establish the theorem of factorization for functions

$$P(x) = 1 + a_1 e^{\alpha_1 x} + \cdots + a_n e^{\alpha_n x},$$

stated in the introduction.

Our first step is to take the polynomial $Q(y_1, \cdots, y_p)$ associated with $P(x)$ in § 5, and to resolve it into irreducible factors with unity for term of lowest degree.

From the irreducible factors of Q which consist of two terms, we obtain the simple factors S of our expression for $P(x)$. Let each y_i be replaced, in these irreducible factors, by its $e^{\mu_i x}$ of § 5. Each factor goes over into a simple function $1+ae^{\alpha x}$. We separate these simple functions into sets such that the α's of the functions of any one set have rational ratios to one another, but have irrational ratios to the α's of any other set. The product of the several functions of each set is a simple function. The simple functions obtained from the several sets form precisely such a set of simple factors $S_1, \cdots, S_s$ of $P(x)$ as is mentioned in the introduction.

We now consider any irreducible factor of Q, say $U(y_1, \cdots, y_r)$, consisting of more than two terms.* Let

$$U(y_1, \cdots, y_r) = V(y_1^{m_1}, \cdots, y_r^{m_r}),$$

with $V(y_1, \cdots, y_r)$ primary. Of course, $V(y_1, \cdots, y_r)$ is irreducible. It gives a factor of $P(x)$ when each y_i is replaced by $e^{m_i \mu_i x}$.

Of all the finite number of polynomials $V(y_1^{t_1}, \cdots, y_r^{t_r})$ whose irreducible factors are primary (§ 8), consider one which has a maximum number, q, of irreducible factors. Let the irreducible factors of the function considered be $V_1, \cdots, V_q$. We say that each V_i gives an irreducible factor of $P(x)$ when each y_j in it is replaced by $e^{m_j \mu_j x_j / t_j}$.

Suppose, for instance, that V_1 does not give an irreducible factor of $P(x)$. Then there must be some $V_1(y_1^{u_1}, \ldots, y_r^{u_r})$ which is reducible. Thus, $V(y^{t_1 u_1}, \cdots, y_r^{t_r u_r})$ has more than q irreducible factors. We may replace each $t_i u_i$ by a submultiple v_i of itself, if necessary, so as to get a polynomial $V(y_1^{v_1}, \cdots, y_r^{v_r})$ with *primary* irreducible factors, greater in number than q.†

* Of course, U need not involve all of the p variables in Q. We are supposing that the $r \leqq p$ variables in U are relettered, if necessary, so as to have the designations $y_1, \cdots, y_r$.

† The irreducible factors of $V(y_1^{t_1 u_1}, \cdots, y_r^{t_r u_r})$ are all obtained from one of them by multiplying the variables by roots of unity. Hence the highest common factor of the exponents of any y_i is the same for all of the irreducible factors. This highest common factor will therefore be a factor of the exponents of y_i in $V(y_1^{t_1 u_1} \cdots, y_r^{t_r u_r})$.

We have thus a contradiction of the assumption that q is a maximum.

When we multiply together the simple factors of $P(x)$ which arise from the binomial factors of Q, and the irreducible factors of $P(x)$ which come from the remaining factors of Q, we have precisely such an expression for $P(x)$ as is described in the statement of our theorem.

It remains to prove the uniqueness of the resolution. It is easy to see that the uniqueness will follow if we can show that if P_1 is a factor of P_2P_3, each P_i being an expression like (1), and if P_1 has no factor in common with P_2, then P_1 is a factor of P_3.

Let

$$P_2P_3 = P_1P_4. \tag{11}$$

There corresponds to (11) a relation among polynomials

$$Q_2Q_3 = Q_1Q_4$$

with Q_1 relatively prime to Q_2. Then Q_3 is divisible by Q_1, so that P_3 is divisible by P_1. The question of uniqueness is thus settled.

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ARITHMETIC OF LOGIC*

BY

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1. **Introduction.** This is probably the first attempt to construct an arithmetic for an algebra of the non-numerical genus.† In his classic treatise, *An Investigation of the Laws of Thought*,‡ Boole developed the thesis that "Logic (is) $\cdots$ a system of processes carried on by the aid of symbols having a definite interpretation, and subject to laws founded on that interpretation alone. But at the same time they exhibit those laws as identical in form with the laws of the general symbols of Algebra, with this simple addition, viz., that the symbols of Logic are further subject to a special law, to which the symbols of quantity as such, are not subject." The special law is what Boole calls the law of duality, $x(1-x)=0$, or the excluded middle; here the indicated multiplication is logical, $1-x$ is the supplement of x. Boole showed therefore that abstractly logic is contained in common algebra.

Taking rational arithmetic $\mathfrak{A}$ ($\equiv$ the theory of numbers in reference to the positive rational integers $0, 1, 2, 3, \cdots$, only) and certain parts of the theory of algebraic numbers, particularly the rudiments of Dedekind's theory of ideals and those of Kronecker's modular systems as our guides, we shall see to what extent Boole's algebra of logic may be *arithmetized* in a precise sense to be defined presently. Although it will be unnecessary to refer explicitly anywhere to the theory of algebraic numbers, it may be mentioned that this theory, which includes rational arithmetic, is a surer guide than the latter in problems of arithmetization. In rational arithmetic the essential abstract structure of the concepts to be extended beyond $0, 1, 2, \cdots$ is often quite ingeniously concealed. This is true, for example, of the G.C.D., L.C.M., and residuation. The theory of ideals, on the other hand, often indicates immediately what transformations by formal equivalence must first be applied to operations or relations of rational arithmetic in order that they shall be significant for sets of elements for which order relations are either irrelevant or meaningless.

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† For the meaning of this term, cf. Whitehead, *Universal Algebra*, p. 29.

‡ London and Cambridge, 1854; reprinted, Chicago, Open Court, 1916, as vol. 2 of Boole's *Collected Logical Works*.

We shall use a few of the commonest notations of algebraic logic; thus $P \supset Q$ for *P implies Q*; $P\colon\equiv\colon Q$ for P, Q, *are formally equivalent*, viz.,

$$P \supset Q.\ Q \supset P,$$

the dot in the last being the logical *and*; $\equiv$ signifies *definitional identity*, except where it occurs in conjunction with mod, when it indicates congruence, as in $\alpha \equiv \beta \bmod \mu$; the special notation $\alpha|\beta$, borrowed from the theory of numbers, where α, β are *classes*, signifies *α contains β*, $\equiv$ *each element of β is in α*. To avoid confusion we shall never use "contains" in relation to arithmetic; conflicting conventions in this respect have already introduced exasperating paradoxes of language into the theory of numbers.

Small Greek letters $\alpha, \beta, \gamma, \cdots$ will always denote classes; $\mathfrak{C}$ is the set of classes discussed; the *null class* is denoted by ω, the *universal class* by ϵ, so that ω, ϵ are the zero, unity of the *algebra of logic* $\mathfrak{L}$. Elements of $\mathfrak{C}$ will be called elements of $\mathfrak{L}$. *Logical* (but not arithmetical) *addition, multiplication of classes* α, β are indicated as usual by $\alpha+\beta$, $\alpha\beta$, and if α is any element of $\mathfrak{L}$, the *supplement* of α is indicated by an accent, α' (instead of the customary bar which is awkward in monotype). Hence α' is the unique* solution of $\alpha+\alpha'=\epsilon$, $\alpha\alpha'=\omega$, where α is any given element of $\mathfrak{L}$. We shall assume a given set $\mathfrak{C}$ of classes and we postulate that if α, β are any elements of $\mathfrak{C}$, then α', $\alpha+\beta$, $\alpha\beta$ are in $\mathfrak{C}$, and, as before, we refer to elements of $\mathfrak{C}$ as elements of $\mathfrak{L}$.

The letters s, p, l, g denote specific operations upon elements of $\mathfrak{L}$, and they are such that (once for all, without further reference) $\alpha t \beta$ for each of $t=s, p, l, g$ is a uniquely determined element of $\mathfrak{L}$; the letters c, d, r denote specific relations such that $\alpha\ t\ \beta$ for each $t=c, d, r$ is uniquely significant in $\mathfrak{L}$. To assist the memory we remark that s, p, l, g, c, d, r may be read *sum, product, least, greatest, congruent, divides, residual*, terms to be defined in the *arithmetic of logic* as opposed to the algebra. The least, greatest here are intended merely to recall classes having with respect to given classes *properties* abstractly identical with those of the G.C.D., L.C.M. with respect to division in $\mathfrak{A}$; these classes g, l are not necessarily the "most" or the "least" inclusive—the rôles with respect to inclusion may be reversed, and g may be either the most or the least inclusive common class of a set, and similarly for l. Hence we shall not use here the names already familiar in Moore's general analysis.

The *zero, unity* in the *arithmetic* of classes will always be denoted by ζ, υ. It will avoid possible confusion if we add that (ζ, υ) are not necessarily equal to (ω, ϵ) respectively.

* For proof of unicity cf. Whitehead, *Universal Algebra*, p. 36.

Suppose all the elements, operations and signs of relations, other than the logical constants, are replaced in a set $\mathfrak{T}$ of propositions by marks without significance beyond that implied by the assertion of the propositions (which include the postulates of the set). Call the result, $C(\mathfrak{T})$, the *content* of $\mathfrak{T}$. If in $C(\mathfrak{T})$ it be possible to assign interpretations to the marks, giving $\mathfrak{T}_j$, such that $\mathfrak{T}_j$ is self-consistent and uniquely significant in terms of the interpretation, we shall call $\mathfrak{T}_j$ an *instance* of $C(\mathfrak{T})$. Sets $\mathfrak{T}_j$ $(j=1, 2, \cdots)$ of propositions having the same content will be called *abstractly identical.*

Our object is to find parts $\mathfrak{A}_j$ of rational arithmetic $\mathfrak{A}$ abstractly identical with parts $\mathfrak{L}_j$ of the algebra of classes, hence also with the algebra of relations, and finally with $\mathfrak{L}$.

In such a project the following type of *invariance under formal equivalence* is extremely useful. Let P_j $(j=1, 2, \cdots)$ be propositions of $\mathfrak{T}$ such that $P_1 : \equiv : P_2$. Then the truth-values of $P_1 \supset P_3$, $P_3 \supset P_1$ are identical with those of $P_2 \supset P_3$, $P_3 \supset P_2$. Thus $: \equiv :$ in propositions is abstractly identical with $=$ in common algebra, and in implications a particular proposition may be replaced by any other which is formally equivalent to it. We shall meet several instances of such transformations which are considerably less obvious than the logic which justifies them. The identical transformation of a set of transformations by formal equivalence is that which replaces each proposition of a given set by itself; any set of transformed propositions (including the set transformed by the identical transformation) is called a *transform* of the original set.

Let $\mathfrak{T}'$ be a transform of $\mathfrak{T}$, and $\mathfrak{A}_j'$ a transform of a part $\mathfrak{A}_j$ of $\mathfrak{A}$. Then if $\mathfrak{T}'$, $\mathfrak{A}_j'$ are abstractly identical, we shall say that $\mathfrak{T}$ is *arithmetized with respect to* $\mathfrak{A}_j$, or simply *arithmetized.*

This kind of arithmetization can be carried much farther for $\mathfrak{L}$ than is done here, but what is given will suffice to show its nature, and it will be evident that at nearly every stage there are alternative ways of proceeding. We shall exhibit arithmetizations of $\mathfrak{L}$ with respect to congruences, the L.C.M., G.C.D., divisibility, primes, and the unique factorization law.

Rational arithmetic $\mathfrak{A}$ presupposes the existence of a special ring. In the whole discussion we shall ignore negative numbers, without loss of generality, as the arithmetic (properties of integers as such) which refers to these can always be thrown back to relations between positive integers only, e.g., as done by Kronecker.

An *abstract ring* $\mathfrak{R}$ is a set $\mathfrak{S}$ of elements $x, y, z, \cdots, u', z', \cdots$, and two operations S, P($\equiv$ *addition*, *multiplication*) which may be performed upon any two equal or distinct elements x, y of $\mathfrak{R}$, in this order, to produce uniquely determined elements xSy, xPy such that the postulates $\mathfrak{R}_j$ $(j=1, 2, 3)$ are satisfied. Elements of $\mathfrak{S}$ will be called elements of $\mathfrak{R}$.

$\mathfrak{R}_1$. If x, y are any two elements of $\mathfrak{R}$, xSy, xPy are uniquely determined elements of $\mathfrak{R}$, and

$$ySx = xSy, \qquad yPx = xPy.$$

$\mathfrak{R}_2$. If x, y, z are any three elements of $\mathfrak{R}$,

$$(xSy)Sz = xS(ySz), \qquad (xPy)Pz = xP(yPz),$$
$$xP(ySz) = (xPy)S(xPz).$$

$\mathfrak{R}_3$. There exist in $\mathfrak{R}$ two distinct* unique elements, denoted by u', z', and called the *unity*, *zero* of $\mathfrak{R}$, such that if x is any element of $\mathfrak{R}$, $xSz'=x$, $xPu'=x$.

These may be compared with the first three postulates of Dickson† for a field, of which they are a transcription, except that the unicity of u', z' is here a postulate, not a theorem, also with Wedderburn's‡ for algebraic fields. In each comparison the omissions are to be particularly noticed. Thus in $\mathfrak{R}$ we can *not* infer $x=y$ from $xPz=yPz$ (this inference is in general false for the special rings $\mathfrak{R}'$ considered later), nor does P have a unique inverse, although we shall later define division. The inference $x=y$ from $xSz=ySz$ also is illegitimate. No attempt has been made in defining $\mathfrak{R}$ to achieve conformity with other definitions of rings; we are concerned only with isolating from fields what is useful for our project.

If S, P and the elements of $\mathfrak{R}$ are specialized by interpretation or by the adjunction to $\mathfrak{R}_j$ $(j=1, 2, 3)$ of further postulates consistent with those for $\mathfrak{R}$, or by both of these restrictions, we shall call the result, $\mathfrak{R}'$, a *special ring*. An instance of $\mathfrak{R}$ is $\mathfrak{A}$.

2. **Algebraic congruence in** $\mathfrak{L}$. Let xCy be a relation in $\mathfrak{R}$ such that, if x, y, z, w are any elements of $\mathfrak{R}$, xCy is uniquely significant in $\mathfrak{R}$ and the postulates (1.1)–(1.4) are satisfied:

(1.1) $$xCy \supset yCx.$$

(1.2) $$xCy \,.\, yCz : \supset : xCz.$$

(1.3) $$xCy \,.\, zCw : \supset : (xSz)C(ySw).$$

(1.4) $$xCy \,.\, zCw : \supset : (xPz)C(yPw).$$

Then C is called *abstract algebraic congruence*.

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* For the theorem, required later, that ζ, υ are distinct, cf. *Principia Mathematica*, 1st edition, p. 231, *24.1. It will not be necessary hereafter to prove that $\mathfrak{L}_j$ is an instance of $\mathfrak{R}$, as $\zeta\neq\upsilon$ is the only proposition not immediately obvious from the definitions.

† *Algebras and their Arithmetics*, p. 201.

‡ Annals of Mathematics, (2), vol. 24 (1923), pp. 237–264, especially p. 240.

If $\mathfrak{R}$ is replaced by its instance $\mathfrak{A}$, an instance of xCy is $aCb \equiv (a \equiv b \bmod m)$, where a, b are integers $\geqq 0$ and m is an integer >0.

In $\mathfrak{A}$ we shall say that C is *arithmetic congruence* if to the instances in $\mathfrak{A}$ of (1.1)–(1.4) be adjoined the three further postulates

$$(1.5) \qquad (a \equiv 0 \mod m) : \equiv : m \text{ divides } a, m \neq 0\ ;$$

$$(1.6) \qquad (ka \equiv kb \mod m) \supset (a \equiv b \mod m'),\ m \neq 0,$$

where $qm' = m$, and $q =$ the G.C.D. of k, m;

$$(1.7) \qquad a \equiv a \bmod m.$$

Any set of propositions in $\mathfrak{L}$ abstractly identical with any transform of (1.1)–(1.7) will, if true, be said to define *arithmetic congruence c in $\mathfrak{L}$*.

As a practical detail we assign by convention the truth value $(+)$ $(\equiv$true$)$ to an asserted proposition, as for example any instance of (1.1), unless it be expressly noted that the value is $(-)$ $(\equiv$false$)$. This merely avoids the repeated assertion that our propositions as stated are $(+)$, which they are.

We shall now proceed to the partial determination of c by solving (1.1)–(1.4) in $\mathfrak{L}$; the discussion in $\mathfrak{L}$ of (1.5), (1.6) must be deferred until after that of the G.C.D. and the residual in $\mathfrak{L}$.

By (1.1) C is symmetric. The only symmetric functions of two classes α, β are (by the laws of tautology and absorption)

$$(2.1) \qquad \alpha\beta,\ \alpha + \beta \text{ and their supplements}$$

$$(2.2) \qquad \alpha' + \beta',\ \alpha'\beta'\ ;$$

$$(2.3) \qquad \alpha'\beta + \alpha\beta' \text{ and its supplement}$$

$$(2.4) \qquad \alpha\beta + \alpha'\beta'.$$

The function (2.3) is that which Daniell* has denoted by $|\alpha - \beta|$ and called the *modular difference* of α, β. This is the naturally suggested function for the solution of our problem. It is interesting therefore that it should be rejected by the mildest of the postulates on C, as may be verified in the same way as done presently for another rejection.

Now, by evident analogies between $\mathfrak{L}$ and the theory of division for Dedekind ideals, also by the concept of congruence with respect to an ideal modulus, and further by the "contains" of Kronecker's modular theory, it is immediately suggested that we introduce an arbitrary class μ, constant in (1.1)–(1.4), and seek inclusion relations between μ and each of the symmetric functions σ in (2.1)–(2.4) to satisfy (1.1)–(1.4).

* Bulletin of the American Mathematical Society, vol. 23 (1916), pp. 446–450.

The only inclusion relations for two classes γ, δ are $\gamma|\delta$, $\gamma \neq \omega$ if $\delta \neq \omega$, and $\delta|\gamma$, $\gamma \neq \epsilon$ if $\delta \neq \epsilon$. We therefore test for each σ the truth of the propositions $\sigma|\mu$, $\mu|\sigma$, either of which may turn out to be (+) or (−). It will be sufficient to attend to (2.1) for brevity. Hence we are to test these propositions for (3.1)–(3.4), the truth values being unknown,

$$\alpha C\beta \equiv \mu \mid \alpha\beta, \qquad \mu \neq \omega \text{ if } \alpha\beta \neq \omega, \tag{3.1}$$

$$\alpha C\beta \equiv \alpha\beta \mid \mu, \qquad \mu \neq \epsilon \text{ if } \alpha\beta \neq \epsilon, \tag{3.2}$$

$$\alpha C\beta \equiv (\alpha + \beta) \mid \mu, \qquad \mu \neq \epsilon \text{ if } \alpha + \beta \neq \epsilon, \tag{3.3}$$

$$\alpha C\beta \equiv \mu \mid (\alpha + \beta), \qquad \mu \neq \omega \text{ if } \alpha + \beta \neq \omega. \tag{3.4}$$

The conclusions are summarized in the following table.

	(1.1)	(1.2)	(1.3)	(1.4)
(3.1)	(+)	(−)	(−)	(+)
(3.2)	(+)	(+)	(+)	(+)
(3.3)	(+)	(−)	(+)	(−)
(3.4)	(+)	(+)	(+)	(+)

It will suffice to verify the row (3.2) and check the falsity of one (−) proposition, say (3.1).(1.3), deferring consideration of the exceptions.

From (3.2) we have $\alpha C\beta \equiv \alpha\beta|\mu$, and (1.1) is (+), as it must be automatically since $\alpha\beta$ is symmetric. For (1.2) in this case we should have (+) for

$$\alpha\beta \mid \mu \,.\, \beta\gamma \mid \mu : \supset : \alpha\gamma \mid \mu.$$

Now (cf. Whitehead, loc. cit., p. 43, prop. 14)

$$\alpha \mid \beta \,.\, \gamma \mid \delta : \supset : \alpha\gamma \mid \beta\delta \,;$$

hence

$$\begin{aligned} \alpha\beta \mid \mu \,.\, \beta\gamma \mid \mu &: \supset : \alpha\beta\beta\gamma \mid \mu\mu, \\ &: \supset : \alpha\beta\gamma \mid \mu \,; \end{aligned}$$

but $\alpha\gamma|\alpha\beta\gamma$; hence $\alpha\gamma|\mu$. Again, (1.3) requires

$$\alpha\beta \mid \mu \,.\, \gamma\delta \mid \mu : \supset : (\alpha + \gamma)(\beta + \delta) \mid \mu,$$

which is (+), since

$$(\alpha + \gamma)(\beta + \delta) \mid \alpha\beta \,.\, (\alpha + \gamma)(\beta + \delta) \mid \gamma\delta \,;$$

and (1.4) in this case is

$$\alpha\beta \mid \mu \,.\, \gamma\delta \mid \mu : \supset : \alpha\beta\gamma\delta \mid \mu$$

which obviously is (+).

Taking the false (3.1).(1.3) we have $\alpha C\beta \equiv \mu | \alpha\beta$, and (1.3) demands

$$\mu \mid \alpha\beta .\ \mu \mid \gamma\delta : \supset : \mu \mid (\alpha + \gamma)(\beta + \delta)$$

which clearly is $(-)$. Similarly for the rest of the table.

Hence we have the alternative *solutions in* $\mathfrak{L}$ *for the problem of algebraic congruence of classes,*

(4.1) $\qquad (\alpha \equiv \beta \bmod \mu) : \equiv : \alpha\beta \mid \mu, \qquad \mu \neq \epsilon,$

(4.2) $\qquad (\alpha \equiv \beta \bmod \mu) : \equiv : \mu \mid (\alpha + \beta), \qquad \mu \neq \omega,$

and evidently either solution can be inferred from the other by the Peirce-Schröder dualism in $\mathfrak{L}$, viz., the reciprocity between logical addition and multiplication. The values of μ which must be excepted in (4.1), (4.2) will be seen later to be abstractly identical with the excepted modulus zero in $\mathfrak{A}$; in each case the *arithmetic* zero ζ is barred as a modulus μ.

Each problem in $\mathfrak{L}$ has a similar two-valued solution. It is economical however to state both duals in each instance in order to decide readily which must be paired from one solution with one from another to yield the required arithmetic applicable to the simultaneous solutions of several postulate systems.

3. **The transform of reflexiveness of congruence.** Examining the solutions (4.1), (4.2) we see that each violates the simplest property of congruence in $\mathfrak{A}$, viz., *reflexiveness*. For in $\mathfrak{A}$ we have the $(+)$ proposition $(1.7) \equiv (5.1)$,

(5.1) $\qquad a \equiv a \bmod m,$

for all elements $a \geqq 0$ of $\mathfrak{A}$ and $m > 0$. But in $\mathfrak{A}$ we have $a - a = 0$. Hence we may replace (5.1) by

(5.2) $\qquad 0 \equiv 0 \bmod m,$

since in $\mathfrak{A}$,

$$(0 \equiv 0 \bmod m) : \equiv : (a \equiv a \bmod m).$$

Hence, comparing (5.2), (1.5), we may replace (5.1) by its transform (5.2), and hence *reflexiveness of congruence in* $\mathfrak{A}$ *may be replaced by the proposition that the zero,* 0, *in* $\mathfrak{A}$ *is divisible by every element of* $\mathfrak{A}$, *with the possible exception* (*removed presently*) *of* "0 *divides* 0."

In $\mathfrak{A}$ we either do not define division by zero, in which case dividends with zero divisors are not in our universe of discourse, or we define division by zero, saying that the quotient is wholly indeterminate, and exclude the process. It is impossible to reconcile either procedure with $\mathfrak{L}$, as will be clear when we come to division in $\mathfrak{L}$, so we make a slight compromise which affects

nothing in $\mathfrak{A}$ but which is necessary in $\mathfrak{L}$. We shall exclude division by zero in $\mathfrak{A}$ except in the one case where the dividend is also zero, and we shall say that in this case the quotient exists but is wholly indeterminate.

4. **Division in** $\mathfrak{L}$. Consider in the abstract ring $\mathfrak{R}$ a relation having the properties

$$xDx, \tag{6.1}$$

$$xDy \,.\, yDz : \supset : xDz, \tag{6.2}$$

$$xDy \,.\, yDx : \supset : x = y, \tag{6.3}$$

where xDy is uniquely significant for each $x \neq z'$ (the zero in $\mathfrak{R}$) and y in $\mathfrak{R}$, with the exception (cf. § 3) that $z'Dz'$ is significant but indeterminate in $\mathfrak{R}$.

These are satisfied in $\mathfrak{A}$ by taking $xDy \equiv x$ *divides* y, and they may be $(+)$ in any instance $\mathfrak{R}'$ of $\mathfrak{R}$ irrespective of whether division yields a unique quotient.* In $\mathfrak{L}$ we shall select a solution which does not give a unique quotient but which does lead to a unique factorization theorem—a rather unexpected situation.

As before, analogy with the theory of ideals suggests that we take in $\mathfrak{L}$ an inclusive relation for D. We shall consider both of

$$\alpha D\beta \equiv \alpha \mid \beta, \tag{7.1}$$

$$\alpha D\beta \equiv \beta \mid \alpha, \tag{7.2}$$

as definitions (in different interpretations) of *algebraic division in* $\mathfrak{L}$. We may read (7.1) as *α divides β, or β is a multiple of α, is identical with α contains β*; (7.2) is read *α divides β, or β is a multiple of α, is identical with β contains α.* Thus (7.1) is as in the theory of ideals; (7.2) is closer to $\mathfrak{A}$.

5. **The G.C.D., L.C.M.** in $\mathfrak{L}$. These afford interesting examples of invariance under formal equivalence. As first defined in $\mathfrak{A}$, the G.C.D. of a, b is the *greatest* integer which divides both a and b; the L.C.M. is the *least* integer which both a and b divide, division as always in $\mathfrak{A}$ being arithmetical, viz., all quotients are required to be in $\mathfrak{A}$. Neither of these is immediately applicable to $\mathfrak{L}$. But they may be replaced by their transforms in $\mathfrak{A}$, precisely as in the theory of ideals: *With every set of elements $a, b, \cdots, h$ of $\mathfrak{A}$ there is associated a unique element m of $\mathfrak{A}$ such that every element of $\mathfrak{A}$ which divides each element in the set divides also m; there also is associated a unique element l such that every element of $\mathfrak{A}$ which is a multiple of each element of the set is also a multiple of l.*

* If for $x \neq z'$ the relation xDy implies the existence in $\mathfrak{R}$ of a unique element w such that $y = xPw$, we say that the *quotient in $\mathfrak{R}$ is unique*

The propositions, if true, abstractly identical with these in any special ring $\mathfrak{R}'$ will be taken as the definitions of the *arithmetic G.C.D. and L.C.M. in* $\mathfrak{R}'$.

Abstracting these propositions to $\mathfrak{R}$, and taking D as in (6.1)–(6.3), we consider two operations G, L upon elements of $\mathfrak{R}$ such that, if x, y are any elements of $\mathfrak{R}$, then xGy and xLy are *uniquely* determined elements of $\mathfrak{R}$, and the postulates (8.1)–(9.4) are satisfied:

(8.1) $$xGy = yGx,$$

(8.2) $$xG(yGz) = (xGy)Gz \equiv xGyGz,$$

(8.3) $$(xGy)Dx \,.\, (xGy)Dy,$$

(8.4) $$zDx \,.\, zDy : \supset : zD(xGy),$$

for G; and for L,

(9.1) $$xLy = yLx,$$

(9.2) $$xL(yLz) = (xLy)Lz \equiv xLyLz,$$

(9.3) $$xD(xLy) \,.\, yD(xLy),$$

(9.4) $$xDz \,.\, yDz : \supset : (xLy)Dz,$$

in all of which x, y, z are any elements of $\mathfrak{R}$.

Note the abstract identity of the pairs (8.1), (9.1) and (8.2), (9.2) with $\mathfrak{R}_1$ and the first of $\mathfrak{R}_2$ in § 1, and observe the interesting reciprocal symmetry between (8.3), (8.4) and (9.3), (9.4).

A solution in $\mathfrak{A}$ of (8.1)–(9.4) is evidently $aGb \equiv$ the G.C.D. of the integers a, $b \geqq 0$, and of (9.1)–(9.4), $aLb \equiv$ the L.C.M. of a, b. Moreover (8.1)–(9.4) together with the postulated unicity of G, L define, or *uniquely determine*, the *arithmetic* G.C.D., L.C.M. of elements of $\mathfrak{A}$. Hence we shall call the solution in $\mathfrak{L}$ of (8.1)–(9.4) the *arithmetic G.C.D. and L.C.M. in* $\mathfrak{L}$, and by taking these properties of the G.C.D. and L.C.M. of *classes* as fundamental, we automatically fix *division and integral elements in* $\mathfrak{L}$.

There are two solutions, according to the choice of D as in either of (7.1), (7.2). The unicity, essential for arithmetic, is obvious in each instance. To indicate that we have now passed from algebra to arithmetic we shall use d, l, g as stated in § 1, instead of D, L, G, and write the definitions

(10.1) $$\alpha d\beta \equiv \alpha \textit{ divides } \beta,$$

(10.2) $$\alpha l\beta \equiv \textit{the L. C. M. of } \alpha, \beta$$

(10.3) $$\alpha g\beta \equiv \textit{the G. C. D. of } \alpha, \beta,$$

(10.4) $$(\zeta, \upsilon) \equiv \textit{the (zero, unity) in } \mathfrak{L}.$$

Assuming for the moment the existence and unicity of ζ, υ, we have the following alternative solutions of the problems of *arithmetic divisibility* (d), *greatest common divisor* (g), *least common multiple* (l) *in* $\mathfrak{L}$:

(11.1) $$\alpha d\beta \equiv \alpha \mid \beta, \quad \alpha g\beta \equiv \alpha + \beta, \quad \alpha l\beta \equiv \alpha\beta,$$

(11.2) $$\alpha d\beta \equiv \beta \mid \alpha, \quad \alpha g\beta \equiv \alpha\beta, \quad \alpha l\beta \equiv \alpha + \beta.$$

It will be of interest to write down the propositions which are the verifications of (8.1)–(9.4). The first column is for (11.1), the second for (11.2):

(8.11) $\alpha + \beta = \beta + \alpha,$ $\qquad \alpha\beta = \beta\alpha,$

(8.21) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma,$ $\qquad \alpha(\beta\gamma) = (\alpha\beta)\gamma,$

(8.31) $(\alpha + \beta) \mid \alpha \,.\, (\alpha + \beta) \mid \beta,$ $\qquad \alpha \mid (\alpha\beta) \,.\, \beta \mid (\alpha\beta),$

(8.41) $\gamma \mid \alpha \,.\, \gamma \mid \beta : \supset : \gamma \mid (\alpha + \beta),$ $\qquad \alpha \mid \gamma \,.\, \beta \mid \gamma : \supset : \alpha\beta \mid \gamma,$

(9.11) $\alpha\beta = \beta\alpha,$ $\qquad \alpha + \beta = \beta + \alpha,$

(9.21) $\alpha(\beta\gamma) = (\alpha\beta)\gamma,$ $\qquad \alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma,$

(9.31) $\alpha \mid (\alpha\beta) \,.\, \beta \mid (\alpha\beta),$ $\qquad (\alpha + \beta) \mid \alpha \,.\, (\alpha + \beta) \mid \gamma,$

(9.41) $\alpha \mid \gamma \,.\, \beta \mid \gamma : \supset : \alpha\beta \mid \gamma,$ $\qquad \gamma \mid \alpha \,.\, \gamma \mid \beta : \supset : \gamma \mid (\alpha + \beta).$

6. **Addition** (s) **and multiplication** (p) **in** $\mathfrak{L}$. A fundamental property in $\mathfrak{A}$ of the G.C.D. and L.C.M. is that the *product* of the G.C.D. and L.C.M. of two elements of $\mathfrak{A}$ is equal to the *product* of the elements. This determines the choice of p and therefore also of s in $\mathfrak{L}$. As before there necessarily are two solutions. Each must (and does) satisfy $\mathfrak{R}_1$, $\mathfrak{R}_2$ of § 1. Writing

(12.1) $$\alpha s\beta \equiv \textit{the arithmetic sum of } \alpha,\beta,$$

(12.2) $$\alpha p\beta \equiv \textit{the arithmetic product of } \alpha,\beta,$$

we see that d, s, p must be paired as follows to preserve the property of g, l just mentioned:

(13.1) $$\alpha d\beta \equiv \alpha \mid \beta, \quad \alpha s\beta \equiv \alpha + \beta, \quad \alpha p\beta \equiv \alpha\beta\,;$$

(13.2) $$\alpha d\beta \equiv \beta \mid \alpha, \quad \alpha s\beta \equiv \alpha\beta, \quad \alpha p\beta \equiv \alpha + \beta.$$

For *each* of the pairs (11.1), (13.1) and (11.2), (13.2) we have

(14) $$(\alpha g\beta)p(\alpha l\beta) = \alpha p\beta.$$

7. **The arithmetic zero** ζ, **unity** υ, **in** $\mathfrak{L}$. The algebraic zero, unity in $\mathfrak{L}$ are ω, ϵ, so that $\alpha+\omega=\alpha$, $\alpha\epsilon=\alpha$. In $\mathfrak{L}$ we must have

(15) $$\alpha s\zeta = \alpha, \qquad \alpha p\upsilon = \alpha,$$

for each element α of $\mathfrak{L}$. Hence in each of (13.1), (13.2), ζ, υ are uniquely determined, and we have

(15.1) $\alpha s\beta \equiv \alpha + \beta, \quad \alpha p\beta \equiv \alpha\beta, \qquad (\zeta, \upsilon) = (\omega, \epsilon),$

(15.2) $\alpha s\beta \equiv \alpha\beta, \quad \alpha p\beta \equiv \alpha + \beta, \quad (\zeta, \upsilon) = (\epsilon, \omega).$

As in $\mathfrak{A}$ the unity in $\mathfrak{L}$ divides each element γ,

(16.1) $\alpha d\beta \equiv \alpha \mid \beta, \quad \upsilon = \epsilon, \quad \upsilon d\gamma;$

(16.2) $\alpha d\beta \equiv \beta \mid \alpha, \quad \upsilon = \omega, \quad \upsilon d\gamma.$

Again, in $\mathfrak{A}$ the only element divisible by the zero in $\mathfrak{A}$ is zero, and in $\mathfrak{L}$ we have, in abstract identity* with $\mathfrak{A}$,

(17.1) $\zeta \mid \gamma \,.\, (\gamma \neq \zeta) \,.\, (\zeta = \omega)$ is $(-)$,

(17.2) $\gamma \mid \zeta \,.\, (\gamma \neq \zeta) \,.\, (\zeta = \epsilon)$ is $(-)$.

8. **Arithmetic congruence in $\mathfrak{L}$.** The proposition in $\mathfrak{L}$ abstractly identical with (1.5) in $\mathfrak{A}$ is

(18) $(\alpha \equiv \zeta \bmod \mu) : \supset : \mu d\alpha,$

which is (+) provided (cf. (4.1), (4.2) and (16.1), (16.2)) we pair as follows the definitions of congruence and divisibility in $\mathfrak{L}$:

(18.1) $\alpha d\beta \equiv \alpha \mid \beta, \qquad (\alpha \equiv \beta \bmod \mu) \equiv \mu \mid (\alpha + \beta),$

(18.2) $\alpha d\beta \equiv \beta \mid \alpha, \qquad (\alpha \equiv \beta \bmod \mu) \equiv \alpha\beta \mid \mu,$

which can be stated together as

(19) $(\alpha \equiv \beta \bmod \mu) \equiv \mu d(\alpha s\beta) \equiv \mu d(\alpha g\beta).$

Since (18) is (+) for each of (18.1), (18.2) it follows that the later are *necessary for arithmetic congruence c in* $\mathfrak{L}$. We have already satisfied (1.7) for c; it remains only to discuss (1.6).

9. **Residuals, completion of c, extremes.** The transformation by formal equivalence of (1.6) in $\mathfrak{A}$ will complete the sequence of properties of arithmetic congruence c in $\mathfrak{L}$ and yield the abstract identity of congruence in $\mathfrak{A}$, $\mathfrak{L}$. The necessary transformation, suggested by the properties of modular systems, is effected by the abstraction of Lasker's† concept of the residual for such systems. We shall first abstract to $\mathfrak{R}$.

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* For $\omega \mid \omega$ cf. *Principia Mathematica*, 1st edition, p. 232, *24.13.

† Mathematische Annalen, vol. 60 (1905), p. 49.

Let a, b, l, m for the moment denote elements of $\mathfrak{R}$. Then, if m is uniquely determined by ($u' \equiv$ the unity in $\mathfrak{R}$),

$$\{aD(lPb)\} \,.\, \{mDl\} \,.\, \{m \neq u'\}, \tag{20}$$

where l runs through all elements in $\mathfrak{R}$, we shall call m the *residual of b with respect to a*, and we shall write $m = bRa$.

In $\mathfrak{L}$, (20) becomes

$$\{\alpha d(\lambda p\beta)\} \,.\, \{\mu d\lambda\} \,.\, \{\mu \neq \upsilon\} : \equiv : \mu \equiv \beta r\alpha, \tag{20.1}$$

where r replaces R in the instance (20.1) of (20), and λ is an arbitrary class.

In $\mathfrak{A}$, the residual of k with respect to m is the quotient of m by the G.C.D. of k and m, viz., this residual is m' in (1.6).

The proposition in $\mathfrak{L}$ abstractly identical with (1.6) is therefore

$$(\kappa p\alpha \equiv \kappa p\beta \bmod \mu) \supset (\alpha \equiv \bmod \kappa r\mu), \tag{21}$$

and this, as may be verified immediately, is implied by (20.1) and p, d, υ as in either of the following columns, which recapitulate previous solutions:

(s)	*Sum*:	$\alpha + \beta$,	$\alpha\beta$,	$\equiv \alpha s\beta$,
(p)	*Product*:	$\alpha\beta$,	$\alpha + \beta$,	$\equiv \alpha p\beta$,
(g)	*G.C.D. of* α, β:	$\alpha + \beta$,	$\alpha\beta$,	$\equiv \alpha g\beta$,
(l)	*L.C.M. of* α, β:	$\alpha\beta$,	$\alpha + \beta$,	$\equiv \alpha l\beta$,
(c)	$\alpha \equiv \beta \bmod \mu$:	$\mu \mid (\alpha + \beta)$,	$\alpha\beta \mid \mu$,	
(ζ)	*Zero*:	ω ,	ϵ ,	$\equiv \zeta$,
(υ)	*Unity*:	ϵ ,	ω ,	$\equiv \upsilon$,
(d)	*α divides β*:	$\alpha \mid \beta$,	$\beta \mid \alpha$,	$\equiv \alpha d\beta$,

the same interpretations for p, d, υ necessarily being taken in both of (20.1), (21).

Either column is implied by the other and the reciprocity between logical addition and multiplication. That $\alpha s\beta = \alpha g\beta$, $\alpha p\beta = \alpha l\beta$ in $\mathfrak{L}$, while the corresponding propositions in $\mathfrak{A}$ are ($-$), is due to the laws of tautology and absorption, but these do not destroy the abstract identity of the arithmetic of logic and rational arithmetic. The identity is in the fundamental propositions, or postulates, stated abstractly as in $\mathfrak{R}$, from which $\mathfrak{A}$ is developed, and we have shown therefore that certain parts of both rational arithmetic and the arithmetic of logic are instances of one and the same *content*. The abstract identity will be enhanced when we find a unique factorization law in $\mathfrak{L}$.

In passing it may be of interest to note the equivalents in $\mathfrak{L}$ of *least*, *greatest* in those parts of $\mathfrak{A}$ which we have abstracted. They are as follows. If in a given set of elements of $\mathfrak{L}$ there be a unique element different from the unity in $\mathfrak{L}$ which divides each element of the set, that element is called the *lower extreme* of the set; if in a given set of elements of $\mathfrak{L}$ there be a unique element different from the zero in $\mathfrak{L}$ which is divisible by each element of the set, that element is called the *upper extreme* of the set. In these definitions either type of division in $\mathfrak{L}$ may be taken; the upper and lower extremes, viz., the classes which these actually are, in either interpretation are inverted in the other. The G.C.D., L.C.M., residual and congruences in $\mathfrak{L}$ can be restated if desired in terms of extremes. If this be done the verbal forms in $\mathfrak{L}$ become the same as those in $\mathfrak{A}$.

10. **Unique factorization in $\mathfrak{L}$.** In $\mathfrak{A}$ a set of elements (integers $\geqq 0$) is said to be coprime if the G.C.D. of all members of the set is unity. Similarly in $\mathfrak{L}$ we define a set of elements to be *coprime* if their G.C.D. is υ. In what follows it is assumed that we are operating in either one of the solutions for $s, p, g, l, c, \zeta, \upsilon, d$ exhibited in § 9; the results hold in either.

For clearness let us recall a few properties of the constituents ($\equiv$terms) of a Boole *development* ($\equiv$expansion*) which will be needed immediately. It is assumed that the development is in *normal form*, viz., that in which all terms with zero coefficients have been deleted. Then first, the logical product of any two distinct terms of a development is the logical zero. Otherwise stated, the terms of a Boole development are a set of classes such that any pair of them are mutually exclusive. The logical sum of all the terms is the logical unity. Hence if α, β denote any two identical (in which case $\beta\equiv\alpha$) or distinct terms of a development, $\alpha|\beta \supset \alpha=\beta$. Second, from a given set of classes we can generate by the operations of logical addition, multiplication and taking of supplements a set closed under these operations; the closed set consists of all the elements of $\mathfrak{L}$. The development of the logical unity of this set provides us with a set of terms such that the development of any element of $\mathfrak{L}$ as a function ($\equiv$logical sum) of such terms is unique. This situation is abstractly identical with the unique factorization theorem in $\mathfrak{A}$, as will be shown in a moment.

Suppose now that we have obtained the unique development as above described of a given element of $\mathfrak{L}$. Since $\mathfrak{L}$ is closed under the operation of taking the supplement, it follows that the development of any element of $\mathfrak{L}$ has a dual, obtained by taking the supplements of both sides of the original

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* *Laws of Thought*, Chapter V, especially Prop. III.

development of the supplement of the given element. This gives us the dual unique decomposition in $\mathfrak{L}$ abstractly identical with the first and with the fundamental theorem of arithmetic.

In translating these properties of $\mathfrak{L}$ to arithmetic it is more intuitive to fix the attention on the second column in § 9. That is, we shall think of

$$\alpha s \beta \equiv \alpha\beta, \qquad \alpha p \beta \equiv \alpha + \beta, \qquad \alpha g \beta \equiv \alpha\beta, \qquad \alpha l \beta \equiv \alpha + \beta,$$
$$\zeta \equiv \epsilon, \qquad \upsilon \equiv \omega, \qquad \alpha d \beta \equiv \beta \mid \alpha,$$

although, by the duality in $\mathfrak{L}$, the theorems are valid in either interpretation. Arranging a few of the abstractly identical theorems and definitions of $\mathfrak{A}$, $\mathfrak{L}$ in pairs, we have the following:

(22.1) If the G.C.D. of a, b is 1, then a, b are called coprime.

(22.2) If the G.C.D. of α, β is υ, then α, β are called coprime.

(23.1) If k divides the product of a and b, and k, a are coprime, then k divides b.

(23.2) $\{\kappa d(\alpha p \beta)\} . (\kappa g \alpha = \upsilon) : \supset : \kappa d \beta.$

(24.1) q is prime if $k \neq 1$ divides q when and only when $k = q$.

(24.2) π is prime if and only if $(\kappa d \pi) . (\kappa \neq \upsilon) : \supset : \kappa = \pi.$

(25.1) Primes exist; they are found by the seive of Eratosthenes and are a coprime set.

(25.2) Primes exist; they are found by the Boole development of ζ and are a coprime set.

(26.1) A positive integer is the product of primes in one way only.

(26.2) A given element of $\mathfrak{L}$ is the arithmetic product (p) of prime elements of $\mathfrak{L}$ in one way only.

This list can obviously be extended; for example we can write down the G.C.D. and L.C.M. of α, β from their resolutions into prime factors in $\mathfrak{L}$ precisely as in $\mathfrak{A}$. Again, abstractly identical with the theory of arithmetical functions in $\mathfrak{A}$, such as the indicator, sum and number of divisors, etc., of an integer, which depend upon the unique factorization law in $\mathfrak{A}$, there is a like theory of functions of classes or relations in $\mathfrak{L}$. Subtraction in this theory is as defined in $\mathfrak{L}$ by Boole. The interpretation in $\mathfrak{L}$ of these arithmetical theorems is however not always an easy matter; its interest here is that arithmetic has reacted upon logic to yield new results in the latter.

If to obtain the elements of $\mathfrak{L}$ we start from a finite set of classes (or relations) we have in the above arithmetic of $\mathfrak{L}$ a finite image of $\mathfrak{A}$. Conversely the theory of inclusion relations in logic is abstractly identical with rational arithmetic.

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FUNCTIONALS OF r-DIMENSIONAL MANIFOLDS ADMITTING CONTINUOUS GROUPS OF POINT TRANSFORMATIONS*

BY

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1. **Introduction.** The study of integral invariants was initiated by H. Poincaré in connection with some important problems in dynamics in his memorable prize memoir‡ on the three-body problem. S. Lie§ showed that the subject of integral invariants is closely connected with his theories of differential invariants of continuous groups of transformations. More recently Goursat and Cartan in taking up the subject from Poincaré's point of view have made important contributions; and in their respective books *Leçons sur le Problème de Pfaff* and *Leçons sur les Invariants Intégraux* they have connected their results with the Pfaffian problem and its generalizations.

A multiple integral extended over an r-dimensional manifold in n dimensions depends for its value on the r-dimensional manifold and its sense,‖ and in general on the parametric representation of the manifold. If one fixes the sense of the manifold and if the multiple integral remains invariant¶ in value for an arbitrary change of parameters, then the multiple integral will depend only on the r-dimensional manifold for its value. In other words, such a multiple integral is a functional of r-dimensional manifolds; and hence Poincaré's and Cartan's integral invariants are additive functional invariants. Taking cognizance of this point of view, one naturally inquires as to the

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‡ Acta Mathematica, vol. 12 (1890). For a more complete exposition see his *Méthodes Nouvelles de la Mécanique Céleste*, vol. III.

§ *Die Theorie der Integralinvarianten ist ein Corollar der Theorie der Differentialinvarianten*, Berichte der Sächsischen Gesellschaft der Wissenschaften, 1897, pp. 342–357.

‖ Cf. Philip Franklin, *Multiple integrals in n-space*, Annals of Mathematics, (2), vol. 24 (1922–1923), pp. 213–226. See also Georges Giraud, *Le problème de Dirichlet généralisé*, Annales de l'École Normale Supérieure, January, 1926.

¶ We note here that some of Lie's work on integral invariants (especially with respect to infinite continuous groups) has reference to such an invariance. Such types of integral invariants are also found in works on tensor analysis. Cf. Weitzenböck, *Invariantentheorie*, chapter XIV; and E. Noether, Göttinger Nachrichten, 1918.

possibility of considering *invariant functionals* which are *not* necessarily *additive. This paper is concerned with such generalizations of the Poincaré and Cartan integral invariant.*

In the first part of the paper we consider first functionals of closed plane curves that have a normal variation and that admit a one-parameter group of the plane. It is found that such invariant functionals are solutions of functional equations with functional derivatives. To get explicit solutions of our problems we assume our functionals to be expansible in a series each term of which is a repeated area integral. It is easy to extend this reasoning to the case of corresponding expansions of functionals of closed $(n-1)$-dimensional manifolds in n dimensions; but the case of functionals of closed space curves requires special considerations. The problem reduces to the solution of a functional equation with functional fluxes. I then indicate briefly what type of functional expansions could be used to solve the equations.

In the second part of the paper certain tensors that depend on i points in space are discussed. Then we consider the invariant theory, under one-parameter groups, of functionals I_i having the form of an i-tuply repeated r-dimensional integral* (see expression (5.4)). In this part of the paper we do not restrict the invariance with respect to closed manifolds only, exception being taken of § 11. In generalizing Goursat's notion of integral invariants attached to the trajectories of a system of differential equations, we are led to a generalization of Cartan's complete integral invariant. One of the most interesting theorems of the paper is that of § 12, which shows the impossibility in general of stepping from a generalized Poincaré integral invariant to a generalized Cartan complete integral invariant.

At the end of the paper certain results are given on functional invariants of s-parameter groups. I propose to make a more complete study of functional invariants of s-parameter groups elsewhere.

There are a number of problems that one may well consider in studying the functional invariants of this paper. For example, the generalization of the Pfaffian problem by means of non-additive functionals offers such a possibility. It is my intention to consider such problems in other papers.

Throughout the whole paper we shall assume without statement, or with slight statement, the necessary continuity, differentiability, analyticity and

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* It is only a matter of calculation to show that the invariant theory of the functional $\sum_{i=0}^{\infty} I_i$ is merely the invariant theory of the representative term. This can be shown by considering an obvious generalization of the lemma of §2. For the sake of brevity we make our calculations only for a representative term I_i.

uniformity properties of our functions in a certain finite region Δ in which we shall work.

Part I

2. **Invariant functionals of closed plane curves.*** We consider a one-parameter continuous group of transformations defined by the system of differential equations

$$\frac{dx}{\xi(M)} = \frac{dy}{\eta(M)} = d\tau. \tag{2.1}$$

In (2.1) M denotes a point in the plane and τ denotes the parameter of the group. We shall assume without any further statement that the functions ξ and η are real analytic functions of x and y in a certain domain Δ. We further assume that ξ and η remain finite in Δ and do not vanish simultaneously in Δ.

Let $\Phi[C]$ be a continuous functional of closed plane curves possessing a normal functional derivative $\Phi_n'[C/M]$, taken at a point M on C, uniformly continuous in C and M. We assume in this paragraph that we are dealing with such functionals $\Phi[C]$ which have a variation of the form

$$\delta\Phi[C] = \int_C \Phi_n'[C/M]\delta n(M)ds. \tag{2.2}$$

(In (2.2) $\delta n(M)$ stands for the variation of the normal to C at the point M on C and ds is the element of arc length of C. We shall make the convention that the outer direction of the normal is positive.

We proceed to prove a lemma which we shall find useful in our subsequent discussion.

Lemma. *If $F[C]$ is a functional of closed curves possessing a development*

$$F[C] = f^{(0)} + \sum_{i=1}^{\infty} \frac{1}{j!} \underbrace{\int_\sigma \int_\sigma \cdots \int_\sigma}_{(i)} f^{(i)}(M_1, M_2, \cdots, M_i) d\sigma_1 d\sigma_2 \cdots d\sigma_i \tag{2.3}$$

(σ is the area within C and $d\sigma_h$ is the element of area at a point M_h) with

$$|f^{(i)}(M_1, M_2, \cdots, M_i)| < K$$

* The analysis given in this paragraph can be extended easily to the case of corresponding invariant problems of functionals of closed $(n-1)$-dimensional manifolds in n dimensions. For an abstract of most of the results of §2 see the Proceedings of the National Academy of Sciences, vol. 11 (1925), pp. 98–101.

and $f^{(i)}(M_1, M_2, \cdots, M_i)$ *symmetric* and continuous in all its i arguments* M_1 *to* M_i, *then a necessary and sufficient condition that*

$$F[C] = 0 \tag{2.4}$$

for all closed curves is that

$$f^{(0)} = 0, \quad f^{(i)}(M_1, M_2, \cdots, M_i) = 0 \quad (i = 1,2,3, \cdots). \tag{2.5}$$

Since the normal functional derivatives of all orders of (2.3) exist and are continuous, it follows from the hypothesis (2.4) that all the functional derivatives of $F[C]$ must vanish. Letting σ shrink indefinitely, we get as a necessary consequence

$$f^{(0)} = 0.$$

Taking the first functional derivative of (2.4) at the point M_1, and letting σ shrink indefinitely we get

$$f^{(1)}(M_1) \equiv 0.$$

On following such a process of functional differentiations followed by letting σ shrink indefinitely, we find

$$f^{(i)}(M_1, M_2, \cdots, M_i) \equiv 0 \qquad (i = 1,2,3, \cdots).$$

A necessary and sufficient condition that a functional $\Phi[C]$ admit the group (2.1) is that its variation vanish identically in virtue of (2.1). Since along the path curves of (2.1) we have

$$\delta n(M) = [\xi(M) \cos (x,n) + \eta(M) \cos (y,n)]d\tau, \tag{2.6}$$

this condition becomes

$$\int_C \Phi_n' [C/M][\xi(M) \cos (x,n) + \eta(M) \cos (y,n)]ds = 0 \tag{2.7}$$

for all closed curves C.

We can write equation (2.7) in the equivalent form

$$\Phi_n' [C/M] = A[C/M] \tag{2.8}$$

where $A[C/M]$ is a given arbitrary functional satisfying the functional equation

$$\int_C A[C/M][\xi(M) \cos (x,n) + \eta(M) \cos (y,n)]\, ds \equiv 0. \tag{2.9}$$

---

* This is no essential restriction, as by a process of symmetrization we can put the functional in such a form.

The integrability conditions* for the functional equation (2.8) can be stated in terms of the notion of an adjoint functional as follows.

A necessary and sufficient condition that (2.8) be integrable is that $\delta A[C/M]$ be a self-adjoint functional of δn. We may proceed now and show that there always exists an infinitude of functionals $A[C/M]$ that satisfy (2.9) and satisfy the integrability conditions for (2.8) identically. Hence there would always be solutions for a Cauchy problem for (2.8). Instead of following such a general argument, however, we shall assume an explicit expansion for $\Phi[C]$ and impose the conditions that such a $\Phi[C]$ satisfy the equation (2.7).

Assume then that we are considering functionals $\Phi[C]$ developable in an expansion

$$\phi^{(0)} + \sum_{i=1}^{\infty} \frac{1}{i!} \underbrace{\int_\sigma \int_\sigma \cdots \int_\sigma}_{(i)} \phi^{(i)}(M_1, M_2, \cdots, M_i) d\sigma_1 d\sigma_2 \cdots d\sigma_i \tag{2.10}$$

where $\phi^{(i)}(M_1, M_2, \cdots, M_i)$ is symmetric in all the i points M_1 to M_i and possesses continuous first partial derivatives, and

$$|\phi^{(i)}| < K.$$

We shall demonstrate the following theorem.

THEOREM 1. *A necessary and sufficient condition that a functional* $\Phi[C]$ *possessing a development* (2.10) *admit the group of transformations defined by* (2.1) *is that each* $\phi^{(i)}(M_1, \cdots, M_i)$ *satisfy the corresponding partial differential equation*

$$\sum_{j=1}^{i} \left\{ \xi(M_j) \frac{\partial \phi^{(i)}(M_1, \cdots, M_i)}{\partial x_j} + \eta(M_j) \frac{\partial \phi^{(i)}(M_1, \cdots, M_i)}{\partial y_j} + \phi^{(i)}(M_1, \cdots, M_i) \left[\frac{\partial \xi(M_j)}{\partial x_j} + \frac{\partial \eta(M_j)}{\partial y_j} \right] \right\} = 0. \tag{2.11}$$

To prove this theorem we make use of the evident relations

$$\begin{aligned} \int_C u(M) \cos(x, n) ds &= \int_\sigma \frac{\partial u}{\partial x} d\sigma, \\ \int_C v(M) \cos(y, n) ds &= \int_\sigma \frac{\partial v}{\partial y} d\sigma. \end{aligned} \tag{2.12}$$

* Cf. G. C. Evans, The Cambridge Colloquium, pp. 20–23.

By a direct calculation* we have

$$\begin{aligned}\Phi_n'[C/M] &= \phi^{(1)}(M)\\ &+ \sum_{i=1}^{\infty}\frac{1}{i!}\int_\sigma\int_\sigma\underset{(i)}{\cdots}\int_\sigma \phi^{(i+1)}(M,M_1,\cdots,M_i)d\sigma_1 d\sigma_2\cdots d\sigma_i.\end{aligned}\tag{2.13}$$

Hence condition (2.7) becomes

$$\begin{aligned}\sum_{i=0}^{\infty}\frac{1}{i!}\int_\sigma\int_\sigma\underset{(i+1)}{\cdots}\int_\sigma\Bigg[&\frac{\partial\phi^{(i+1)}(M,M_1,\cdots,M_i)\xi(M)}{\partial x}\\ &+\frac{\partial\phi^{(i+1)}(M,M_1,\cdots,M_i)\eta(M)}{\partial y}\Bigg]d\sigma\, d\sigma_1 d\sigma_2\cdots d\sigma_i = 0\end{aligned}\tag{2.14}$$

for all closed curves C.

On symmetrizing the integrands without changing the values of the integrations and on renaming the subscripts, condition (2.14) becomes

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{1}{i!}\int_\sigma\int_\sigma\underset{(i)}{\cdots}\int_\sigma\Bigg[\sum_{j=1}^{i}\Bigg\{&\frac{\partial\xi(M_j)\phi^{(i)}(M_1,\cdots,M_i)}{\partial x_j}\\ &+\frac{\partial\eta(M_j)\phi^{(i)}(M_1,\cdots,M_i)}{\partial y_j}\Bigg\}\Bigg]d\sigma_1\cdots d\sigma_i = 0.\end{aligned}\tag{2.15}$$

Hence by our lemma, a necessary and sufficient condition that (2.15) hold is that†

$$\sum_{j=1}^{i}\left[\frac{\partial\xi(M_j)\phi^{(i)}(M_1,\cdots,M_i)}{\partial x_j}+\frac{\partial\eta(M_j)\phi^{(i)}(M_1,\cdots,M_i)}{\partial y_j}\right] = 0 \qquad (i = 1,2,\cdots).\tag{2.16}$$

Our theorem therefore follows readily.

From the form of the equations (2.11) it is clear that if $\psi(M_1, \cdots, M_i)$ is a solution then $\psi(M_l, \cdots, M_r)$ is also a solution, where $M_l, \cdots, M_r$ is any permutation of $M_1, \cdots, M_i$. Hence there always exist symmetric functions of $M_1, \cdots, M_i$ that satisfy equation (2.11). Thus there always exist functionals of closed plane curves of form (2.10) admitting a given arbitrary continuous one-parameter group of transformations (2.1).

* Functional differentiations and integrations term by term are valid since the series involved are uniformlyt convergent.

† We note hat this condition affirms the invariance of each term of (2.10).

We shall prove the following theorem which will give information as to the most general functional invariant of type (2.10).

THEOREM 2. *If*

$$(2.17)\quad \Phi[C] = \phi_0^{(0)} + \sum_{i=1}^{\infty} \frac{1}{i!} \int_\sigma \int_\sigma \underset{(i)}{\cdots} \int_\sigma \phi_0^{(i)}(M_1, \cdots, M_i) d\sigma_1 \cdots d\sigma_i$$

is a functional admitting the group of transformations (2.1) *and if* $I^{(i)}(M_1, \cdots, M_i)$ *is the most general symmetric function of* $M_1, \cdots, M_i$ *which admits the ith cograediently extended group of transformations* (2.1), *then the most general functional of type* (2.10) *admitting the group of transformations* (2.1) *can be written as*

$$(2.18)\quad \begin{aligned} \Phi[C] &= \phi_{00}^{(0)} \\ &+ \sum_{i=1}^{\infty} \frac{1}{i!} \int_\sigma \int_\sigma \underset{(i)}{\cdots} \int_\sigma I^{(i)}(M_1, \cdots, M_i)\phi_0^{(i)}(M_1, \cdots, M_i) d\sigma_1 \cdots d\sigma_i \end{aligned}$$

($\phi_{00}^{(0)}$ *being an arbitrary constant*).

Let $\psi^{(i)}(M_1, \cdots, M_i)$ be the representative integrand in the most general functional invariant; then by our hypothesis the following equations are satisfied:

$$(2.19)\quad \begin{aligned} &\sum_{j=1}^{i} \left[\xi(M_j) \frac{\partial \phi_0^{(i)}}{\partial x_j} + \eta(M_j) \frac{\partial \phi_0^{(i)}}{\partial y_j} + \phi_0^{(i)} \left\{ \frac{\partial \xi(M_j)}{\partial x_j} + \frac{\partial \eta(M_j)}{\partial y_j} \right\} \right] = 0, \\ &\sum_{j=1}^{i} \left[\xi(M_j) \frac{\partial \psi^{(i)}}{\partial x_j} + \eta(M_j) \frac{\partial \psi^{(i)}}{\partial y_j} + \psi^{(i)} \left\{ \frac{\partial \xi(M_j)}{\partial x_j} + \frac{\partial \eta(M_j)}{\partial y_j} \right\} \right] = 0. \end{aligned}$$

Hence it follows that $\psi^{(i)}/\phi_0^{(i)}$ is an invariant function of the ith cogrediently extended group (2.1). Conversely, one can verify readily that if $\psi^{(i)}/\phi_0^{(i)}$ is an invariant then the functions $\psi^{(i)}$ will form an invariant functional. The theorem follows therefore readily.

In order to get necessary and sufficient conditions for the invariance of a functional derivative we shall need to study the invariance of functionals $\Phi[C/M]$ depending in particular on a point* M on the curve.

We assume that the variation of a functional $\Phi[C/M]$ has the form

$$(2.20)\quad \delta\Phi[C/M] = \frac{\partial \Phi[C/M]}{\partial x}\delta x + \frac{\partial \Phi[C/M]}{\partial y}\delta y + \int_C \Phi'_{n_1}[C/M, M_1]\delta n(M_1) ds_1.$$

* The point M of course being itself subject to the groups of transformations (2.1).

A necessary and sufficient condition that a functional $\Phi[C/M]$ with a variation of the form (2.20) admit the group of transformations (2.1) is that $\Phi[C/M]$ satisfy the functional equation

$$
\begin{aligned}
(2.21)\qquad &\frac{\partial\Phi[C/M]}{\partial x}\xi(M) + \frac{\partial\Phi[C/M]}{\partial y}\eta(M) \\
&+ \int_C \Phi'_{n_1}[C/M, M_1][\xi(M_1)\cos(x_1, n_1) + \eta(M_1)\cos(y_1, n_1)]ds_1 = 0.
\end{aligned}
$$

For an explicit solution of our problem we assume $\Phi[C/M]$ to be developable in an expansion

$$
\begin{aligned}
(2.22)\qquad \Phi[C/M] &= \phi^{(0)}(M) \\
&+ \sum_{i=1}^{\infty}\frac{1}{i!}\int_\sigma\int_\sigma \overset{(i)}{\cdots} \int_\sigma \phi^{(i)}(M, M_1, \cdots, M_i)d\sigma_1\cdots d\sigma_i,
\end{aligned}
$$

where $\phi^{(i)}(M, M_1, \cdots, M_i)$ is symmetric in the points $M_1, M_2, \cdots, M_i$; possesses continuous first partial derivatives; and

$$|\phi^{(i)}| < K.$$

On going through a similar analysis to the one used in deriving conditions for the invariance of a functional $\Phi[C]$ of type (2.10) the truth of the following theorem can be verified.

THEOREM 3. *A necessary and sufficient condition that a functional $\Phi[C/M]$ possessing a development* (2.22) *admit the group of transformations defined by* (2.1) *is that each $\phi^{(i)}(M, M_1, \cdots, M_i)$ satisfy the corresponding partial differential equation*

$$
\begin{aligned}
&\xi(M)\frac{\partial\phi^{(i)}(M, M_1, \cdots, M_i)}{\partial x} + \eta(M)\frac{\partial\phi^{(i)}(M, M_1, \cdots, M_i)}{\partial y} \\
(2.23)\quad &+ \sum_{j=1}^{i}\Bigg[\xi(M_j)\frac{\partial\phi^{(i)}(M, M_1, \cdots, M_i)}{\partial x_j} + \eta(M_j)\frac{\partial\phi^{(i)}(M, M_1, \cdots, M_i)}{\partial y_j} \\
&+ \phi^{(i)}(M\ M_1. \cdots, M_i)\left\{\frac{\partial\xi(M_j)}{\partial x_j} + \frac{\partial\eta(M_j)}{\partial y_j}\right\}\Bigg] = 0(i = 0, 1, 2, \cdots).
\end{aligned}
$$

On comparing the conditions for the invariance of a functional $\Phi[C]$ of type (2.10) and its normal functional derivatives* $\Phi'_n[C/M]$ we find that the following conditions have to be imposed on the group (2.1):

* Such a functional will have the form (2.22) in which all the integrands will be symmetric functions of *all* their point arguments.

$$\frac{\partial \xi(M)}{\partial x} + \frac{\partial \eta(M)}{\partial y} = 0. \tag{2.24}$$

Hence a necessary and sufficient condition that the normal functional derivative $\Phi_n'[C/M]$ of a functional $\Phi[C]$ with a development (2.10) which admits the group (2.1) be itself invariant of (2.1) is that the group (2.1) be area preserving.

3. **Invariant functionals of closed space curves.** The theory given in the preceding paragraph, although directly extensible to functionals of closed surfaces in space, is not amenable to a discussion of functionals of closed *curves* in space. We shall see, however, that the invariant theory of a certain large class of functionals of closed space curves involves the invariant theory of functionals of closed plane curves as a special case.

Let $F_x'[C/t]$, $F_y'[C/t]$, $F_z'[C/t]$ be the functional derivatives along the x, y and z directions respectively of a functional $F[C]$ of closed space curves C. These functional derivatives are taken at a point M on the curve C having t as the corresponding value of the parameter that defines the position of a point on the curve.

We shall deal with functionals whose variation* is given by

$$\delta F[C] = \int_C \{F_x'[C/t]\delta x(t) + F_y'[C/t]\delta y(t) + F_z'[C/t]\delta z(t)\}\,dt. \tag{3.1}$$

From the well known identity

$$F_x'dx + F_y'dy + F_z'dz = 0 \tag{3.2}$$

that holds along the curves C, Volterra deduced the existence of a functional quantity V (called the flux of the functional $F[C]$ by Paul Lévy) with components V_x, V_y, V_z.

In terms of the functional flux V,† the variation (3.1) takes the form

$$\delta F[C] = \int_C \begin{vmatrix} dx & dy & dz \\ \delta x & \delta y & \delta z \\ V_x & V_y & V_z \end{vmatrix}. \tag{3.3}$$

Consider a one-parameter group of transformations

$$\frac{dx}{\xi(M)} = \frac{dy}{\eta(M)} = \frac{dz}{\zeta(M)} = d\tau. \tag{3.4}$$

---

* Cf. Volterra, Acta Mathematica, vol. 12, pp. 237–244. See also Evans, The Cambridge Colloquium, Part I, pp. 8–12, and Hadamard, *Leçons sur le Calcul des Variations*, pp. 286–287.

† It is easy to see that the functional flux is essentially an alternating covariant tensor.

A necessary and sufficient condition that a functional $F[C]$ of closed space curves admit a one-parameter group (3.4) is that its functional flux V satisfy the functional equation

$$\int_C \begin{vmatrix} dx & dy & dz \\ \xi & \eta & \zeta \\ V_x & V_y & V_z \end{vmatrix} = 0. \tag{3.5}$$

We note that in the special case in which the curves C are plane, we can take the xy plane as that of the curves C. Then if we take one-parameter groups (2.1) in that plane, we are led immediately to the functional equation (2.7).

To get an explicit solution of the functional equation (3.5) we can assume that our functional $F[C]$ is of the form

$$\phi_0 + \sum_{i=1}^{\infty} \frac{1}{i!} \underbrace{\int_\sigma \int_\sigma \cdots \int_\sigma}_{(i)} \phi_{n_1 n_2 \cdots n_i}(M_1, M_2, \cdots, M_i) d\sigma_1 d\sigma_2 \cdots d\sigma_i. \tag{3.6}$$

In (3.6), ϕ_0 is a constant;

$$\begin{aligned}
\phi_{n_1}(M_1) &= \psi_{x_1}(M_1) \cos(x_1, n_1) + \psi_{y_1}(M_1) \cos(y_1, n_1) \\
&\quad + \psi_{z_1}(M_1) \cos(z_1, n_1), \\
\phi_{n_1 n_2}(M_1, M_2) &= \psi_{x_1 x_2}(M_1, M_2) \cos(x_1, n_1) \cdot \cos(x_2, n_2) \\
&\quad + \psi_{x_1 y_2}(M_1, M_2) \cos(x_1, n_1) \cdot \cos(y_2, n_2) \\
&\quad + \psi_{x_1 z_2}(M_1, M_2) \cos(x_1, n_1) \cdot \cos(z_2, n_2) \\
&\quad + \psi_{y_1 x_2}(M_1, M_2) \cos(y_1, n_1) \cdot \cos(x_2, n_2) \\
&\quad + \psi_{y_1 y_2}(M_1, M_2) \cos(y_1, n_1) \cdot \cos(y_2, n_2) \\
&\quad + \psi_{y_1 z_2}(M_1, M_2) \cos(y_1, n_1) \cdot \cos(z_2, n_2) \\
&\quad + \psi_{z_1 x_2}(M_1, M_2) \cos(z_1, n_1) \cdot \cos(x_2, n_2) \\
&\quad + \psi_{z_1 y_2}(M_1, M_2) \cos(z_1, n_1) \cdot \cos(y_2, n_2) \\
&\quad + \psi_{z_1 z_2}(M_1, M_2) \cos(z_1, n_1) \cdot \cos(z_2, n_2),
\end{aligned} \tag{3.7}$$

. . . .
. . . .

Furthermore, σ is the area of a surface capping the closed curve C. The functional (3.6) will depend only on the closed curve C if*

* Cf. Cornelia Fabri, *Sopra alcune proprietà generali delle funzioni che dipendono da altre funzioni e da linee*, Atti della Reale Accademia delle Scienze di Torino, vol. 25, pp. 671.

$$
(3.8)\qquad
\begin{aligned}
&\frac{\partial\psi_{x_1}}{\partial x_1}+\frac{\partial\psi_{y_1}}{\partial y_1}+\frac{\partial\psi_{z_1}}{\partial z_1}=0,\\
&\frac{\partial\psi_{x_1x_2}}{\partial x_2}+\frac{\partial\psi_{x_1y_2}}{\partial y_2}+\frac{\partial\psi_{x_1z_2}}{\partial z_2}=0,\\
&\frac{\partial\psi_{y_1x_2}}{\partial x_2}+\frac{\partial\psi_{y_1y_2}}{\partial y_2}+\frac{\partial\psi_{y_1z_2}}{\partial z_2}=0,\\
&\frac{\partial\psi_{z_1x_2}}{\partial x_2}+\frac{\partial\psi_{z_1y_2}}{\partial y_2}+\frac{\partial\psi_{z_1z_2}}{\partial z_2}=0,\\
&\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots.
\end{aligned}
$$

In (3.6) we have assumed, without any loss of generality, that

$$
(3.9)\qquad
\begin{aligned}
\psi_{x_1x_2}(M_1,M_2)&=\psi_{x_1x_2}(M_2,M_1),\\
\psi_{y_1y_2}(M_1,M_2)&=\psi_{y_1y_2}(M_2,M_1),\\
\psi_{z_1z_2}(M_1,M_2)&=\psi_{z_1z_2}(M_2,M_1),\\
\psi_{x_1y_2}(M_1,M_2)&=\psi_{y_1x_2}(M_2,M_1),\\
\psi_{x_1z_2}(M_1,M_2)&=\psi_{z_1x_2}(M_2,M_1),\\
\psi_{y_1z_2}(M_1,M_2)&=\psi_{z_1y_2}(M_2,M_1),\\
&\cdots\cdots\cdots\\
&\cdots\cdots\cdots\\
&\cdots\cdots\cdots.
\end{aligned}
$$

We are now in a position to solve the functional equation (3.5) by methods analogous to those given in § 2. The detailed derivation of conditions corresponding to (2.11) entails lengthy calculations. It is here that the tensor analysis enters most appropriately. In fact for functionals* of closed r-dimensional manifolds in n dimensions, tensor considerations are a practical necessity.

PART II

4. **Tensors whose components depend on i points.** In most textbooks and papers dealing with tensor analysis a tensor is defined as a set of quantities having the following transformation law† under an arbitrary

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* Cf. §11, Part II of this paper.

† From now on a repetition of a *Greek letter index* in a term will indicate summation from 1 to n unless otherwise stated.

analytic coördinate transformation (x) to $(\bar{x})$ with non-vanishing jacobian:

$$\overline{T}^{\alpha\beta\cdots\epsilon}_{\lambda\mu\cdots\sigma}(\bar{x}) = T^{\zeta\eta\cdots\kappa}_{\nu\phi\cdots\psi}(x)\frac{\partial x^\nu}{\partial \bar{x}^\lambda}\frac{\partial x^\phi}{\partial \bar{x}^\mu}\cdots\frac{\partial x^\psi}{\partial \bar{x}^\sigma}\frac{\partial \bar{x}^\alpha}{\partial x^\zeta}\frac{\partial \bar{x}^\beta}{\partial x^\eta}\cdots\frac{\partial \bar{x}^\epsilon}{\partial x^\kappa}. \tag{4.1}$$

Each component of the tensor T in (4.1) is assumed in general to be a point function. More recently Paul Dienes* has studied tensors whose components depend also on a curve in connection with the non-integrable cases of Levi-Civita's equations of infinitesimal parallelism. More generally, there are tensors whose components depend also on r-dimensional manifolds in n dimensions. Such for example is the case in the flux† of a non-additive functional of closed r-dimensional manifolds in n dimensions. Such a flux is essentially a covariant tensor of rank $r+1$ whose components are functionals of the closed r-dimensional manifolds depending on a variable point on the manifold.

In the functional invariant theories which I propose to develop now, there occur expressions whose components are functions of i points of the *same* n-dimensional space. Let $(x_k^1, x_k^2, \cdots, x_k^n)$ be the coördinates of a point M_k in a coördinate system x of an n-dimensional space. Consider an analytic transformation of coördinates x to $\bar{x}$ with non-vanishing jacobian. A set of quantities

$$T^{\zeta\cdots\eta,\theta\cdots\kappa,\cdots,\nu\cdots\sigma}_{\alpha\cdots\beta,\gamma\cdots\delta,\cdots,\lambda\cdots\mu}(M_1, M_2, \cdots, M_i) \tag{4.2}$$

whose law of transformation under the ith cogrediently extended group of analytic transformations is

$$\overline{T}^{f\cdots g,h\cdots k,\cdots,p\cdots s}_{a\cdots b,c\cdots d,\cdots,l\cdots m}(\overline{M}_1, \overline{M}_2, \cdots, \overline{M}_i) \tag{4.3}$$

$$= T^{\zeta\cdots\eta,\theta\cdots\kappa,\cdots,\nu\cdots\sigma}_{\alpha\cdots\beta,\gamma\cdots\delta,\cdots,\lambda\cdots\mu}(M_1, M_2, \cdots, M_i)\frac{\partial x_1^\alpha}{\partial \bar{x}_1^a}\cdots\frac{\partial x_1^\beta}{\partial \bar{x}_1^b}\frac{\partial x_2^\gamma}{\partial \bar{x}_2^c}\cdots\frac{\partial x_2^\delta}{\partial \bar{x}_2^d}$$

$$\cdots\frac{\partial x_i^\lambda}{\partial \bar{x}_i^l}\cdots\frac{\partial x_i^\mu}{\partial \bar{x}_i^m}\frac{\partial \bar{x}_1^b}{\partial x_1^\zeta}\cdots\frac{\partial \bar{x}_1^g}{\partial x_1^\eta}\cdots\frac{\partial \bar{x}_i^p}{\partial x_i^\nu}\cdots\frac{\partial \bar{x}_i^s}{\partial x_i^\sigma},$$

---

* Paul Dienes, *Sur la structure mathématique du calcul tensoriel*, Journal de Mathématiques, 1924.

† Volterra's work on functionals of r-dimensional manifolds can be considerably simplified and enriched by such a tensor point of view. I hope to take up this study in later papers.

will be termed a *multiple tensor**. We shall speak of the covariant or contravariant rank of such a multiple tensor with respect to each point M_k. For example, the multiple tensor (4.2) with respect to the point M_2 is covariant with a rank equal to the number of indices γ to δ and contravariant with a rank equal to the number of indices θ to κ.

It is clear from our definition of a multiple tensor that most of the laws of the ordinary tensor algebra can be extended to our case. For example, we shall find useful the following three properties:

(I) If all the components of a multiple tensor vanish identically in a particular coördinate system, they vanish identically in every coördinate system.

(II) If the relation

$$
\begin{aligned}
&T_{\alpha_j\cdots\beta_j,\cdots,\gamma_p\cdots\delta_p,\cdots,\lambda_s\cdots\mu_s,\cdots,\nu_l\cdots\sigma_l}(M_j,\cdots,M_p,\cdots,M_s,\cdots,M_l) \\
(4.4)\quad &= T_{\alpha_j\cdots\beta_j,\cdots,\lambda_s\cdots\mu_s,\cdots,\gamma_p\cdots\delta_p,\cdots,\nu_l\cdots\sigma_l}(M_j,\cdots,M_s,\cdots,M_p,\cdots,M_l)
\end{aligned}
$$

holds in one coördinate system, it holds in every coördinate system.

(III) If a multiple tensor T is alternating *separately* in *each set* of subscripts $\alpha_j\cdots\beta_j$ to $\nu_l\cdots\sigma_l$ in one coördinate system, it is in every coördinate system.

Certain multiple tensors will appear frequently in our work, so to escape digression in the coming paragraphs it will be advantageous to discuss them presently. We define the four sets of quantities

$$
\begin{aligned}
&R_{\alpha_1\beta_1\cdots\sigma_1,\cdots,\alpha_k\beta_k\cdots\sigma_k\omega,\cdots,\alpha_i\cdots\sigma_i}(M_1,\cdots,M_k,\cdots,M_i), \\
&\overset{(k)}{R}_{\alpha_1\cdots\sigma_1,\cdots,\alpha_i\cdots\sigma_i}(M_1,\cdots,M_i), \\
&S_{\alpha_1\cdots\rho_1\sigma_1,\cdots,\alpha_k\cdots\rho_k *,\cdots,\alpha_i\cdots\rho_i\sigma_i}(M_1,\cdots,M_k,\cdots,M_i), \\
&\overset{(k)}{S}_{\alpha_1\cdots\sigma_1,\cdots,\alpha_i\cdots\sigma_i}(M_1,\cdots,M_i)
\end{aligned}
$$

in terms of a contravariant vector $X^\omega(M)$ and a multiple tensor $T_{\alpha_1\cdots\sigma_1,\cdots,\alpha_i\cdots\sigma_i}(M_1,\cdots,M_i)$ in the following manner:

$$
\begin{aligned}
&R_{\alpha_1\beta_1\cdots\rho_1\sigma_1,\cdots,\alpha_k\beta_k\cdots\rho_k\sigma_k\omega,\cdots,\alpha_i\beta_i\cdots\rho_i\sigma_i}(M_1,\cdots,M_k,\cdots,M_i) \\
&= \frac{\partial T_{\alpha_1\beta_1\cdots\sigma_1,\cdots,\alpha_k\beta_k\cdots\sigma_k,\cdots,\alpha_i\cdots\sigma_i}}{\partial x_k^\omega} - \frac{\partial T_{\alpha_1\cdots\sigma_1,\cdots,\omega\beta_k\cdots\rho_k\sigma_k,\cdots,\alpha_i\cdots\sigma_i}}{\partial x_k^{\alpha_k}} \\
&\qquad - \cdots - \frac{\partial T_{\alpha_1\cdots\sigma_1,\cdots,\alpha_k\beta_k\cdots\rho_k\omega,\cdots,\alpha_i\cdots\sigma_i}}{\partial x_k^{\sigma_k}},
\end{aligned}
$$

---

* It is obvious that such a multiple tensor can be considered as a particular tensor (in the ordinary sense) in an associated manifold of $i\,n$ dimensions with respect to a certain special group of transformations. Such a point of view, however, is not convenient for our paper.

$$
\begin{aligned}
&R^{(k)}_{\alpha_1\cdots\sigma_1,\cdots,\alpha_i\cdots\sigma}(M_1,\cdots,M_i)\\
&\qquad = X^{\omega}_{(k)}R_{\alpha_1\cdots\sigma_1,\cdots,\alpha_k\cdots\sigma_k\omega,\cdots,\alpha_i\cdots\sigma_i}(M_1,\cdots,M_k,\cdots,M_i),\\
&S_{\alpha_1\cdots\rho_1\sigma_1,\cdots,\alpha_k\cdots\rho_k *,\cdots,\alpha_i\cdots\rho_i\sigma_i}(M_1,\cdots,M_k,\cdots,M_i)\\
&\qquad = X^{\omega}_{(k)}T_{\alpha_1\cdots\rho_1\sigma_1,\cdots,\alpha_k\cdots\rho_k\omega,\cdots,\alpha_i\cdots\rho_i\sigma_i}(M_1,\cdots,M_k,\cdots,M_i),\\
&S^{(k)}_{\alpha_1\cdots\sigma_1,\cdots,\alpha_i\cdots\sigma_i}(M_1,\cdots,M_i)\\
&\qquad = \frac{\partial S_{\alpha_1\cdots\sigma_1,\cdots,\alpha_k\cdots\rho_k *,\cdots,\alpha_i\cdots\sigma_i}}{\partial x_k^{\sigma_k}} - \frac{\partial S_{\alpha_1\cdots\sigma_1,\cdots,\sigma_k\beta_k *\cdots\rho_k,\cdots,\alpha_i\cdots\sigma_i}}{\partial x_k^{\alpha_k}}\\
&\qquad\qquad - \cdots - \frac{\partial S_{\alpha_1\cdots\sigma_1,\cdots,\alpha_k\cdots\pi_k\sigma_k *,\cdots,\alpha_i\cdots\sigma_i}}{\partial x_k^{\rho_k}}\\
&\qquad\qquad\qquad (k = 1,2,\cdots,i).
\end{aligned} \tag{4.5}
$$

In (4.5) we have written $X^{\omega}_{(k)}$ as a convenient notation for $X^{\omega}(M_k)$.

If the multiple tensor $T_{\alpha_1\cdots\sigma_1,\cdots,\alpha_i\cdots\sigma_i}$ is alternating separately in each of the i sets of subscripts $\alpha_1\cdots\sigma_1$ to $\alpha_i\cdots\sigma_i$ and if it satisfies symmetry relations of type (4.4) then the four sets of quantities defined in (4.5) are multiple tensors with corresponding alternating and symmetry relations. This statement follows from a direct calculation of the law of transformation of these four sets of quantities from the known transformation law of the multiple tensor T and the contravariant vector X. Furthermore, we note that the symbol $\star$ in $S_{\alpha_1\cdots\sigma_1,\cdots,\alpha_k\cdots\rho_k *,\cdots,\alpha_i\cdots\sigma_i}$ can be considered as on equal footing with the subscripts α_k to ρ_k with respect to the alternating character of S.

5. **Functionals involving multiple covariant tensors.** Let

$$x^i = f^i(u_1,u_2,\cdots,u_r) \qquad (i=1,2,\cdots,n) \tag{5.1}$$

be the equations of an r-dimensional manifold S_r and D_r the image of S_r on the space of the parameters u. As we shall have to deal with functions of several points in space we shall use the notation

$$x_k^i = f^i(u_{k1},u_{k2},\cdots,u_{kr}) \tag{5.2}$$

to mean the coördinates of a point M_k of the manifold S_r.

Let

$$\delta_{lm}x_p^{\alpha} = \frac{\partial x_p^{\alpha}}{\partial u_{lm}}\delta u_{lm} \qquad (p,l = 1,2,\cdots,i\,;m = 1,2,\cdots .r) \tag{5.3}$$

and in the case of a one-dimensional manifold we shall write

$$\delta_l x_p^a = \frac{\partial x_p^a}{\partial u_l} \delta u_l.$$

Consider the following functional of an r-dimensional manifold S_r:

$$(5.4)\qquad \underbrace{\int_{D_r} \int_{D_r} \cdots \int_{D_r}}_{(i)} T_{\alpha_1\beta_1\cdots\sigma_1,\alpha_2\beta_2\cdots\sigma_2,\cdots,\alpha_i\beta_i\cdots\sigma_i}(M_1, M_2,$$

$$\cdots, M_i)\delta_{11}x_1^{\alpha_1}\delta_{12}x_1^{\beta_1}\cdots\delta_{1r}x_1^{\sigma_1}\delta_{21}x_2^{\alpha_2}\cdots\delta_{2r}x_2^{\sigma_2}\cdots\delta_{i1}x_i^{\alpha_i}\cdots\delta_{ir}x_i^{\sigma_i}.$$

In (5.4) the multiple tensor $T_{\alpha_1\cdots\sigma_1,\cdots,\alpha\cdots\sigma_i}$ is assumed to be alternating in each *set* of variables $\alpha_1\cdots\rho_1$ to $\alpha_i\cdots\rho_i$. Thus the value of (5.4) does not depend on any particular choice of the parameters u but is indeed only dependent on an r-dimensional manifold S_r. We shall *assume* also, *without any loss of generality**, that the *multiple covariant tensor* $T_{\alpha_1\cdots\sigma_1,\cdots,\alpha_i\cdots\sigma_i}$ *satisfies* the symmetry relation of type (4.4). For brevity in writing, we shall sometimes use the notation

$$(5.5)\qquad \int_{D_r}^{(i)} T(i)\delta x$$

for a functional of type (5.4).

6. **Some preliminary material.** In n dimensions we consider the one-parameter continuous group of transformations defined by

$$(6.1)\qquad \frac{dx^i}{d\tau} = X^i(M) \qquad (i = 1, 2, \cdots, n).$$

In (6.1) we assume that the components of the vector X^i are analytic functions of their arguments in a certain finite region in which we shall work.

Unless otherwise stated, a total differentiation with respect to the parameter τ will denote differentiation along the path curves of the group (6.1). Clearly, if the parameters u in (5.4) are taken to be independent of the group parameter τ, we shall have the following system of equations:†

---

* Since the domain of integration D_r is the same in each of the i integrations, it follows that the functional can be put in such a form without altering its value.

† Equations (6.2) form a system which is the ith cogrediently extended variational equations of Poincaré connected with the system of equations (6.1).

$$(6.2)\qquad \begin{aligned} \frac{d(\delta_{lm}x_p^\alpha)}{d\tau} &= \frac{\partial X_{(p)}^\alpha}{\partial x_p^\beta}\delta_{lm}x_p^\beta, \\ \frac{d(\delta_l x_p^\alpha)}{d\tau} &= \frac{\partial X_{(p)}^\alpha}{\partial x_p^\beta}\delta_l x_p^\beta \quad (p,l = 1,2,\cdots,i; m = 1,2,\cdots,r), \end{aligned}$$

where we have written $X_{(p)}^\alpha$ for $X^\alpha(M_p)$.

If $A_{\alpha_1\cdots\sigma_1,\cdots,\alpha_i\cdots\sigma_i}(M_1,\cdots,M_i)$ is a multiple covariant tensor, we deduce, in virtue of (6.1), the following relations:

$$(6.3)\qquad \frac{dA_{\alpha_1\cdots\sigma_1,\cdots,\alpha_i\cdots\sigma_i}(M_1,\cdots,M_i)}{d\tau} = \frac{\partial A_{\alpha_1\cdots\sigma_1,\cdots,\alpha_i\cdots\sigma_i}(M_1,\cdots,M_i)}{\partial x_\gamma^\beta}X_{(\gamma)}^\beta,$$

where γ is summed from 1 to i.

7. **Invariant functionals of one-dimensional manifolds.** The functional identities that arise in the invariant theory of functionals of type (5.4) for $r=1$ (one-dimensional manifolds) require a separate treatment from those that arise in the cases $r>1$. In fact the discussion of the case $r=1$ has to be given in two parts. The reason for this will become evident presently.

Let us consider then quadratic functionals (5.4) of one-dimensional manifolds S_1; i.e., functionals of type

$$(7.1)\qquad \int_{u_0}^{u}\int_{u_0}^{u} T_{\alpha,\beta}(M_1,M_2)\delta_1x_1^\alpha\delta_2x_2^\beta\,.$$

In order that the functional (7.1) admit the group of transformations (6.1) it is necessary and sufficient that the total derivative of (7.1) with respect to the parameter τ be zero for all one-dimensional manifolds. This statement is merely the analytic condition for the following geometrical situation:

Let $\tau=0$ correspond to the position of the one-dimensional manifold S_1. Consider the transform $S_1(\tau)$ of S_1 by the finite transformations of the group (6.1). In general, a functional (7.1) extended over the image $D_1(\tau)$ of $S_1(\tau)$ will depend on the group parameter τ. Since the parameters u are chosen independent of τ we have

$$\int_{D_1(\tau)}^{(2)}[T(2)\delta x]_\tau = \int_{D_1}^{(2)}[T(2)\delta x]_\tau,$$

where $[T(2)\delta x]_\tau$ is the transform of $T(2)\delta x$ by the finite transformations of the group (6.1). The condition

$$\int_{D_1}^{(2)} T(2)\delta x = \int_{D_1}^{(2)}[T(2)\delta x]_\tau$$

for all values of τ, and that irrespective of the initial one-dimensional manifold S_1, states the invariance of the functional (7.1) with respect to the one-parameter group defined by (6.1).

Again since the parameter u is chosen independent of τ, the condition for invariance becomes

$$\int_{u_0}^{u}\int_{u_0}^{u}\frac{d}{d\tau}\left(T_{\alpha,\beta}(M_1,M_2)\delta_1 x_1^\alpha \delta_2 x_2^\beta\right) = 0. \tag{7.2}$$

By (6.1), (6.2), (6.3) and by a rearrangement of the umbral symbols condition (7.2) becomes

$$\int_{u_0}^{u}\int_{u_0}^{u}\left(\frac{\partial T_{\alpha,\beta}}{\partial x_\lambda^\gamma}X^\gamma_{(\lambda)} + T_{\gamma,\beta}\frac{\partial X^\gamma_{(1)}}{\partial x_1^\alpha} + T_{\alpha,\gamma}\frac{\partial X^\gamma_{(2)}}{\partial x_2^\beta}\right)\delta_1 x_1^\alpha \delta_2 x_2^\beta = 0, \tag{7.3}$$

where λ is umbral with range 1 to 2. Since $T_{\alpha,\beta}(M_1, M_2) = T_{\beta,\alpha}(M_2, M_1)$, we see that the parenthesis in (7.3) is also a multiple tensor $F_{\alpha,\beta}(M_1, M_2)$ having the property

$$F_{\alpha,\beta}(M_1,M_2) = F_{\beta,\alpha}(M_2,M_1). \tag{7.4}$$

Equation (7.3) is of the form*

$$\int_{u_0}^{u}\int_{u_0}^{u}\phi(u_1,u_2)\delta u_1 \delta u_2 = 0 \tag{7.5}$$

in which $\phi(u_1, u_2)$ is a symmetric function of u_1 and u_2. By a double use of Leibnitz' formula, once in differentiating partially with respect to u and another time with respect to u_0, we get

$$\phi(u_1,u_2) = 0. \tag{7.6}$$

Hence a necessary and sufficient condition that (7.3) hold is that

$$\left(\frac{\partial T_{\alpha,\beta}(M_1,M_2)}{\partial x_\lambda^\gamma}X^\gamma_{(\lambda)} + T_{\gamma,\beta}(M_1,M_2)\frac{\partial X^\gamma_{(1)}}{\partial x_1^\alpha}\right.$$
$$\left. + T_{\alpha,\gamma}(M_1,M_2)\frac{\partial X^\gamma_{(2)}}{\partial x_2^\beta}\right)\delta_1 x_1 \delta_2^\alpha x_2^\beta = 0 \tag{7.7}$$

(λ umbral with range 1 to 2)

for all points M_1 and M_2 on any arbitrarily given one-dimensional manifold. If we momentarily fix our attention on two arbitrarily given points M_1 and

* For a given set of functions x the form of a one-dimensional manifold is determined but not its extent. This then means in our problem that for a given set of functions x of the parameter u, the interval (u_0, u) in the one-dimensional u-space is arbitrary since the extent of the manifold S is arbitrary.

M_2, and consider all one-dimensional manifolds passing through M_1 and M_2, we see that the infinitesimal displacements $\delta_1 x_1^\alpha$ and $\delta_2 x_2^\beta$ are arbitrary among themselves and are independent of the x's, since the direction of the manifold at M_1 and M_2 is arbitrary.

From (7.7) then we have the system of equations

$$\frac{\partial T_{\alpha,\beta}(M_1,M_2)}{\partial x_\lambda^\gamma} X_\lambda^\gamma + T_{\gamma,\beta}(M_1,M_2)\frac{\partial X_{(1)}^\gamma}{\partial x_1^\alpha} + T_{\alpha,\gamma}(M_1,M_2)\frac{\partial X_{(2)}^\gamma}{\partial x_2^\beta} = 0 \tag{7.8}$$

$(\alpha,\beta = 1,2, \cdots, n$; and λ summed from 1 to 2).

We shall rewrite (7.8) in a form which will make evident the covariant character of (7.8) by making use of the quantities defined in (4.5). We thus have the conditions

$$\sum_{\lambda=1}^{2}[\overset{(\lambda)}{R}_{\alpha,\beta}(M_1,M_2) + \overset{(\lambda)}{S}_{\alpha,\beta}(M_1,M_2)] = 0 \qquad (\alpha,\beta = 1,2, \cdots, n). \tag{7.9}$$

Since each of the quantities $\overset{(1)}{R}_{\alpha,\beta}, \overset{(2)}{R}_{\alpha,\beta}, \overset{(1)}{S}_{\alpha,\beta}, \overset{(2)}{S}_{\alpha,\beta}$ is a multiple covariant tensor of rank one in M_1 and rank one in M_2, it follows that (7.9) is a multiple tensor equation covariant of rank one in M_1 and of rank one in M_2.

Thus we have the following theorem:

THEOREM 1. *A necessary and sufficient condition that the quadratic functional* (7.1) *admit the one-parameter group* (6.1) *is that the components of the multiple tensor* $T_{\alpha,\beta}$ *satisfy* the tensor equation* (7.9).

We turn now to a discussion of the invariance of functionals $F_i[S_1]$ of one-dimensional manifolds having the form

$$\int_{u_0}^{u}\int_{u_0}^{u}\cdot\overset{(i)}{\cdot}\cdot\int_{u_0}^{u} T_{\alpha,\beta,\cdots,\iota}(M_1,M_2, \cdots ,M_i)\delta_1 x_1^\alpha \delta_2 x_2^\beta \cdots \delta_{i-1}x_{i-1}^\theta \delta_i x_i^\iota \qquad (i > 2). \tag{7.10}$$

A necessary and sufficient condition that (7.10) admit the group of transformations (6.1) is that

$$\frac{d}{d\tau}\int_{D_1}^{(i)} T(i)\delta x = 0 \tag{7.11}$$

for all one-dimensional manifolds S, and hence for all one-dimensional images D_1.

* The existence and character of the solutions of the system of equations (7.9) will be discussed later. Cf. the discussion for equations (7.17).

By (6.1), (6.2) and (6.3) and by a rearrangement of the umbral symbols condition (7.11) becomes

$$(7.12)\quad \int_{u_0}^{u}\int_{u_0}^{u}\underset{(i)}{\cdots}\int_{u_0}^{u}\left(\frac{\partial T_{\alpha,\beta,\cdots,\iota}}{\partial x_\lambda^\omega}X_{(\lambda)}^{\omega}+T_{\omega,\beta,\cdots,\iota}\frac{\partial X_{(1)}^{\omega}}{\partial x_1^\alpha}\right.$$
$$\left.+T_{\alpha,\omega,\gamma,\cdots,\iota}\frac{\partial X_{(2)}^{\omega}}{\partial x_2^\beta}+\cdots+T_{\alpha,\beta,\cdots,\theta,\omega}\frac{\partial X_{(i)}^{\omega}}{\partial x_i^\iota}\right)\delta_1x_1^\alpha\delta_2x_2^\beta\cdots\delta_ix_i^\iota=0$$

(λ umbral with range 1 to i).

Since the multiple tensor $T_{\alpha,\beta,\cdots,\iota}$ has the symmetry properties of type (4.4) it follows that the parenthesis in (7.12) also has the same symmetry relations. Hence (7.12) is an equation of the form

$$(7.13)\quad \int_{u_0}^{u}\int_{u_0}^{u}\underset{(i)}{\cdots}\int_{u_0}^{u}\phi(u_1,u_2,\cdots,u_i)\,\delta u_1\delta u_2\cdots\delta u_i=0 \qquad (i>2),$$

where ϕ is a symmetric function of all its i arguments.

A *sufficient** condition that (7.13) hold is that

$$(7.14)\quad \phi(u_1,u_2,\cdots,u_i)=0.$$

Hence a sufficient condition that (7.12) hold is that

$$(7.15)\quad \left(\frac{\partial T_{\alpha,\beta,\cdots,\iota}}{\partial x_\lambda^\omega}X_{(\lambda)}^{\omega}+T_{\omega,\beta,\cdots,\iota}\frac{\partial X_{(1)}^{\omega}}{\partial x_1^\alpha}+\cdots+T_{\alpha,\beta,\cdots,\theta,\omega}\frac{\partial X_{(i)}^{\omega}}{\partial x_i^\iota}\right)$$
$$\times\,\delta_1x_1^\alpha\delta_2x_2^\beta\cdots\delta_ix_i^\iota=0 \qquad (\lambda \text{ umbral with range 1 to } i).$$

Since the direction of the manifold at fixed but arbitrary points M_1, $M_2,\cdots,M_i$ is arbitrary we must have

$$(7.16)\quad \frac{\partial T_{\alpha,\beta,\cdots,\iota}}{\partial x_\lambda^\omega}X_{(\lambda)}^{\omega}+T_{\omega,\beta,\cdots,\iota}\frac{\partial X_{(1)}^{\omega}}{\partial x_1^\alpha}+\cdots+T_{\alpha,\beta,\cdots,\theta,\omega}\frac{\partial X_{(i)}^{\omega}}{\partial x_i^\iota}=0$$

($\alpha,\beta,\cdots,\iota=1,2,\cdots,n$; and λ umbral with range 1 to i).

We can write condition (7.16) in the form

$$(7.17)\quad \sum_{\lambda=1}^{i}\left[R^{(\lambda)}_{\alpha,\beta,\cdots,\theta,\iota}(M_1,M_2,\cdots,M_i)+S^{(\lambda)}_{\alpha,\beta,\cdots,\theta,\iota}(M_1,M_2,\cdots,M_i)\right]=0$$

($\alpha,\beta,\cdots,\iota=1,2,\cdots,n$)

in terms of the multiple tensors defined in (4.5).

---

* It appears that *necessary* and sufficient conditions for the condition (7.13) are of a complicated sort.

Since each $R^{(\lambda)}$ and $S^{(\lambda)}$ in (7.17) is a multiple tensor covariant of rank one in each of the points $M_1, M_2, \cdots, M_i$, it follows that (7.17) is a multiple tensor equation covariant of rank one in each of the points $M_1, M_2, \cdots, M_i$.

It is known* that by a suitable change of coördinate system, say x to $\bar{x}$, the contravariant vector X of the group (6.1) can be made to have components $(0, 0, \cdots, 0, 1)$ in the new coördinate system $\bar{x}$. In fact, by taking a transformation of coördinates†

$$\bar{x}^i = f^i(x_1, x_2, \cdots, x^n) \qquad (i = 1,2, \cdots, n) \tag{7.18}$$

where

$$f^j(x^1, x^2, \cdots, x^n) = c_j \qquad (j = 1,2, \cdots, n-1)$$

are $n-1$ functionally independent first integrals of the system of equations (6.1) and

$$f^n(x^1, x^2, \cdots, x^n) = \tau + c_n$$

is a last integral of (6.1), we see that (6.1) will take the form

$$\frac{d\bar{x}^1}{0} = \frac{d\bar{x}^2}{0} = \cdots = \frac{d\bar{x}^{n-1}}{0} = \frac{d\bar{x}^n}{1} = d\tau. \tag{7.19}$$

Consequently in this new canonical coördinate system $\bar{x}$, the tensor equations (7.17) state that each component of the multiple tensor $\bar{T}_{\alpha,\beta,\cdots,\iota}(\bar{M}_1, \cdots, \bar{M}_i)$ must satisfy the partial differential equation

$$\sum_{\lambda=1}^{i} \frac{\partial f(\bar{M}_1, \bar{M}_2, \cdots, \bar{M}_i)}{\partial \bar{x}_\lambda^n} = 0. \tag{7.20}$$

The following theorem therefore follows readily.

THEOREM 2. *A sufficient condition (in the case $i=2$, both a necessary and sufficient condition) that a functional of type* (7.10) *admit the group of transformations* (6.1) *is that the components of the multiple tensor $T_{\alpha,\beta,\cdots,\iota}$ in the canonical coördinate system $\bar{x}$ be functions of the i cogredient sets of $n-1$ first integrals, and of the differences of the cogredient canonical variables $\bar{x}_\lambda^n$; i.e., $\bar{T}$ has the form*

$$\bar{T}_{\alpha,\beta,\cdots,\iota}(\bar{x}_1^1, \cdots, \bar{x}_1^{n-1}, \bar{x}_1^n - \bar{x}_2^n, \bar{x}_2^1, \cdots, \bar{x}_2^{n-1}, \bar{x}_2^n - \bar{x}_3^n, \cdots, \bar{x}_{i-1}^n - \bar{x}_i^n, \bar{x}_i^1, \cdots, \bar{x}_i^{n-1}).$$

* This is merely a statement in the language of tensor algebra of the well known fact that all one-parameter groups are similar to a one-parameter group of translations.

† Cf. Goursat, *Leçons sur le Problème de Pfaff*, §55.

8. **Invariant functionals of r-dimensional manifolds.*** We turn now to the invariant theory of functionals having the form (5.4). A necessary and sufficient condition that (5.4) admit the group (6.1) is that along the path curves of (6.1)

$$\frac{d}{d\tau}\int_{D_r}^{(i)} T(i)\delta x = 0 \tag{8.1}$$

for all r-dimensional manifolds.

Calculating the condition (8.1) on employing the ith cogrediently extended group of (6.1) and using the relations (5.3), (6.2) and (6.3) we get the condition

$$\begin{aligned}
&\int_{D_r}\underset{(i)}{\int_{Dr}\cdots}\int_{D_r}\Bigg(\frac{\partial T_{\alpha_1\beta_1\cdots\sigma_1,\alpha_2\cdots\sigma_2,\cdots,\alpha_i\cdots\sigma_i}}{\partial x_\lambda^\omega}\overset{\omega}{X}_{(\lambda)} + T_{\omega\beta_1\cdots\sigma_1,\cdots,\alpha_i\cdots\sigma_i}\frac{\partial \overset{\omega}{X}_{(1)}}{\partial x_1^{\alpha_1}} \\
&\quad + \cdots + T_{\alpha_1\cdots\rho_1\omega,\cdots,\alpha_i\cdots\sigma_i}\frac{\partial \overset{\omega}{X}_{(1)}}{\partial x_1^{\sigma_1}} + \cdots + T_{\alpha_1\cdots\sigma_1,\cdots,\alpha_i\cdots\rho_i}\frac{\partial \overset{\omega}{X}_{(i)}}{\partial x_i^{\sigma_i}}\Bigg) \\
&\quad\times \delta_{11}x_1^{\alpha_1}\delta_{12}x_1^{\beta_1}\cdots\delta_{1r}x_1^{\sigma_1}\cdots\delta_{i1}x_i^{\alpha_i}\cdots\delta_{ir}x_i^{\sigma_i} = 0
\end{aligned} \tag{8.2}$$

(λ umbral with range 1 to i).

For a given arbitrary set of functions x, equation (8.2) is of the form

$$\int_{D_r}\int_{D_{r_{(i)}}}\cdots\int_{D_r}\phi(u_{11},u_{12},\cdots,u_{1r};u_{21},\cdots,u_{2r};\cdots;u_{i1},\cdots,u_{ir})\,\delta u_{11}\cdots\delta u_{ir} = 0 \tag{8.3}$$

for arbitrary images D_r in the space of the parameters u. In (8.3) ϕ is symmetric in the i *sets* of u's. Hence by an obvious extension of the fundamental lemma of §2, we must have $\phi=0$ in the parameters u for any arbitrary given set of functions x.

By an evident extension of the corresponding argument in the preceding paragraph it follows that the parenthesis expression in (8.2) must vanish. Employing the multiple tensors defined in §4, we can write this condition in the following form:

$$\begin{aligned}
&\sum_{\lambda=1}^{i}\Big[\overset{(\lambda)}{R}_{\alpha_1\beta_1\cdots\sigma_1,\alpha_2\cdots\sigma_2,\cdots,\alpha_i\cdots\sigma_i}(M_1,M_2,\cdots,M_i) \\
&\quad + \overset{(\lambda)}{S}_{\alpha_1\beta_1\cdots\sigma_1,\alpha_2\cdots\sigma_2,\cdots,\alpha_i\cdots\sigma_i}(M_1,M_2,\cdots,M_i)\Big] = 0
\end{aligned} \tag{8.4}$$

$(\alpha_1,\cdots,\sigma_i = 1,2,\cdots,n)$.

* In this paragraph the *necessary* and sufficient conditions will have reference to the cases $r>1$ and $i=1, 2, \cdots$; or to the case $r=1$, $i=2$.

The multiple tensor character of equations (8.4) follows from the fact that each $R^{(\lambda)}$ and $S^{(\lambda)}$ is a multiple tensor.

To demonstrate the existence of solutions for the partial differential equations (8.4) and to exhibit the character of these solutions, we consider the tensor equations (8.4) in the coördinate system $\bar{x}$ defined by (7.18). In this special coördinate system $\bar{x}$, conditions (8.4) state that each component of $\overline{T}_{\alpha_1 \cdots \sigma_1, \cdots, \alpha_i \cdots \sigma_i}(\overline{M}_1, \cdots, \overline{M}_i)$ will have to satisfy the same partial differential equation (7.20). We can therefore state the following theorem.

THEOREM 3. *A necessary and sufficient condition that a functional* (5.4) *of r-dimensional manifolds admit the group of transformations* (6.1) *is that the components of the multiple tensor* $T_{\alpha_1 \cdots \sigma_1, \cdots, \alpha_i \cdots \sigma_i}$ *in the canonical coördinate system* $\bar{x}$ *be functions of the i cogredient sets of* $n-1$ *first integrals* $\bar{x}_\lambda^\alpha$ $(\alpha=1, \cdots, n-1)$ *and of the differences* $\bar{x}_{\lambda-1}^n - \bar{x}_\lambda^n$ $(\lambda=2, \cdots, i)$.

9. Invariant functionals attached to the path curves of a group. Let $\lambda(M)$ be a real analytic function of $x^1, x^2, \cdots, x^n$. Consider the one-parameter group whose infinitesimal transformations* are

$$\frac{dx^i}{d\tau} = \lambda(M)X^i(M) \qquad (i = 1,2, \cdots, n). \tag{9.1}$$

We lay down the following definition.

DEFINITION.† *A functional admitting the infinitesimal transformations* (9.1) *for all functions* $\lambda(M)$ *will be called an invariant functional attached to the path curves of the one-parameter group* (6.1).

We shall demonstrate the following theorem.

---

* It is clear that the path curves of the group of transformations (9.1) are the same as those of (6.1) for any function $\lambda(M)$.

† This definition is a generalization of Goursat's definition of integral invariants attached to the trajectories of a system of differential equations (6.1). Cf. Goursat's *Leçons sur le Problème de Pfaff*, pp. 236. It is clear from our definition that an invariant functional attached to the path curves of group (6.1) can be considered as a functional which is invariant under the system of differential equations

$$\frac{dx^1}{X^1} = \frac{dx^2}{X^2} = \cdots = \frac{dx^n}{X^n}.$$

In other words, then, we are here generalizing certain Cartan invariant theories. For example, equations (9.2) in differential form are the generalizations of characteristic systems of differential forms. Cf. Cartan, *Leçons sur les Invariants Intégraux*, Chapter IV.

Theorem 1. *Necessary and sufficient conditions that a functional of r-dimensional manifolds* (5.4) *be an invariant functional attached to the path curves of a one-parameter group of transformations* (6.1) *are that the following multiple tensor equations hold*:

$$
(9.2)\quad
\begin{array}{ll}
\text{(a)} & R^{(\lambda)}_{\alpha_1\cdots\sigma_1,\cdots,\alpha_i\cdots\sigma_i}(M_1,\cdots,M_i)=0,\\
\text{(b)} & S_{\alpha_1\cdots\rho_1\sigma_1,\cdots,\alpha_k\cdots\rho_k *,\cdots,\alpha_i\cdots\rho_i\sigma_i}(M_1,\cdots,M_k,\cdots,M_i)=0
\end{array}
$$

$$(\alpha_1,\cdots,\sigma_i=1,2,\cdots,n;\ k,\lambda=1,2,\cdots,i).$$

Necessary and sufficient conditions that a functional (5.4) be an invariant functional attached to the path curves of (6.1) are that conditions (8.4) hold for all functions $\lambda(M)$ when $\lambda(M)X^{\alpha}(M)$ is substituted for $X^{\alpha}(M)$ in (8.4). On rearranging the terms this condition takes the form

$$
(9.3)\quad
\begin{aligned}
&\sum_{\mu=1}^{i}\lambda(M_\mu)\Big[R^{(\mu)}_{\alpha_1\cdots\sigma_1,\cdots,\alpha_i\cdots\sigma_i}(M_1,\cdots,M_i)\\
&\qquad+S^{(\mu)}_{\alpha_1\cdots\sigma_1,\cdots,\alpha_i\cdots\sigma_i}(M_1,\cdots,M_i)\Big]\\
&+S_{\alpha_1\beta_1\cdots\rho_1 *,\cdots,\alpha_i\cdots\sigma_i}\frac{\partial\lambda(M_1)}{\partial x_1^{\sigma_1}}-S_{\sigma_1\beta_1\cdots\rho_1 *,\cdots,\alpha_i\cdots\sigma_i}\frac{\partial\lambda(M_1)}{\partial x_1^{\alpha_1}}\\
&-\cdots-S_{\alpha_1\beta_1\cdots\pi_1\sigma_1 *,\cdots,\alpha_i\cdots\sigma_i}\frac{\partial\lambda(M_1)}{\partial x_1^{\rho_1}}+\cdots\\
&+S_{\alpha_1\cdots\sigma_1,\cdots,\alpha_i\beta_i\cdots\rho_i *}\frac{\partial\lambda(M_i)}{\partial x_i^{\sigma_i}}-S_{\alpha_1\cdots\sigma_1,\cdots,\sigma_i\beta_i\cdots\rho_i *}\frac{\partial\lambda(M_i)}{\partial x_i^{\alpha_i}}\\
&-\cdots-S_{\alpha_1\cdots\sigma_1,\cdots,\alpha_i\beta_i\cdots\pi_i\sigma_i *}\frac{\partial\lambda(M_i)}{\partial x_i^{\rho_i}}=0.
\end{aligned}
$$

Since $S_{\cdots\cdots *\cdots\cdot}$ is an alternating multiple tensor separately in the i sets of subscripts and since $\lambda(M)$ is an arbitrary function, conditions (9.3) yield the conditions

$$R^{(\mu)}_{\alpha_1\cdots\sigma_1,\cdots,\alpha_i\cdots\sigma_i}(M_1,\cdots,M_i)+S^{(\mu)}_{\alpha_1\cdots\sigma_1,\cdots,\alpha_i\cdots\sigma_i}(M_1,\cdots,M_i)=0$$

$$(\alpha_1,\cdots,\sigma_i=1,2,\cdots,n;\ \mu=1,2,\cdots,i)$$

and conditions (b) of (9.2).

But in virtue of the conditions (b) of (9.2), we have

$$S^{(\mu)}_{\alpha_1\cdots\sigma_1,\cdots,\alpha_i\cdots\sigma_i}(M_1,\cdots,M_i)=0$$

$$(\alpha_1,\cdots,\sigma_i=1,2,\cdots,n;\ \mu=1,2,\cdots,i).$$

The theorem follows therefore readily.

The multiple tensor equations (9.2) in the canonical coördinate system $\bar{x}$ defined by (7.18) take the simple form

$$
\frac{\partial \bar{T}_{\alpha_1 \cdots \sigma_1, \cdots, \alpha_i \cdots \sigma_i}(\bar{M}_1, \cdots, \bar{M}_i)}{\partial \bar{x}_\mu^n} = 0, \tag{9.4}
$$

$$
[\bar{T}_{\alpha_1 \cdots \sigma_1, \cdots, \alpha_i \cdots \sigma_i}(\bar{M}_1, \cdots, \bar{M}_i)]_n = 0
$$

$$
(\alpha_1, \cdots, \sigma_i = 1, 2, \cdots, n; \mu = 1, 2, \cdots, i),
$$

where $[\bar{T}_{\alpha_1 \cdots \sigma_1, \cdots, \alpha_i \cdots \sigma_i}(\bar{M}_1, \cdots, \bar{M}_i)]_n$ stands for any one of the components of the multiple tensor $\bar{T}_{\alpha_1 \cdots \sigma_1, \cdots, \alpha_i \cdots \sigma_i}$ in which at least one of the subscripts α_1 to σ_i has the value n. Hence we have proved the following theorem.

THEOREM 2. *A necessary and sufficient condition that a functional* (5.4) *of r-dimensional manifolds be an invariant functional attached to the path curves of group* (6.1) *is that the differential form in* (5.4) *considered in the canonical coördinate system* (7.18), *be expressible only in terms of the i cogredient sets of the $n-1$ first integrals $\bar{x}_\mu^\alpha$ $(\alpha = 1, 2, \cdots, n-1)$ and their differentials.*

It is interesting to note that the conditions for non-additive functionals are more stringent than for the additive ones in that the first set of conditions (9.4) for the additive case $(i=1)$ state that the functional is an integral invariant of (6.1) while in the case of the non-additive functionals $(i>1)$ a special invariance is implied.

We shall now give a method by means of which we shall be able to derive an invariant functional of $(r-1)$-dimensional manifolds attached to the path curves of (6.1) from a special invariant functional of r-dimensional manifolds that is not attached to the path curves of (6.1).

Let

$$
\int_{D_r}^{(i)} T(i)\delta x \tag{9.5}
$$

be an invariant functional of r-dimensional manifolds that satisfies the special conditions

$$
R^{(\lambda)}_{\alpha_1 \cdots \sigma_1, \cdots, \alpha_i \cdots \sigma_i}(M_1, \cdots, M_i) + S^{(\lambda)}_{\alpha_1 \cdots \sigma_1, \cdots, \alpha_i \cdots \sigma_i}(M_1, \cdots, M_i) = 0 \tag{9.6}
$$

$$
(\alpha_1, \cdots, \sigma_i = 1, 2, \cdots, n; \lambda = 1, 2, \cdots, i)
$$

and is not attached to the path curves of (6.1). We shall show* that the functional of $(r-1)$-dimensional manifolds

$$\int_{D_{r-1}}^{(i)} T(i)\delta x \tag{9.7}$$

is an invariant functional attached to the path curve of (6.1) when the relation between the $T(i)$ in (9.7) and the $T(i)$ in (9.5) is given by

$$\begin{aligned} &T_{\alpha_1\cdots\rho_1,\cdots,\alpha_i\cdots\rho_i}(M,\cdots,M_i) \\ &\quad = T_{\alpha_1\cdots\rho_1\sigma_1,\cdots,\alpha_i\cdots\rho_i\sigma_i}(M_1,\cdots,M_i)X^{\sigma_1}_{(1)}\cdots X^{\sigma_i}_{(i)}. \end{aligned} \tag{9.8}$$

This assertion is proved briefly by observing that in the canonical coordinate $\bar{x}$ given by (7.18) the multiple tensor

$$\overline{T}_{\alpha_1\cdots\rho_1,\cdots,\alpha_i\cdots\rho_i}\ (\overline{M}_1,\cdots,\ \overline{M}_i)$$

satisfies conditions of the form (9.4).

In the theory of integral invariants† one finds the following theorem:

A necessary and sufficient condition that an integral invariant be an integral invariant that is attached to the path curves of (6.1) *is that its value be identically zero for all r-dimensional manifolds generated by the path curves of* (6.1).

Such a theorem, however, does not go through, in general, for the non-additive functionals that we have been considering. The necessity part of the theorem is obviously satisfied. The sufficiency part, however, does not hold; for, the identical vanishing of

$$\int_{P_r}\int_{P_r}\cdots\int_{P_r} T_{\alpha_1\cdots\rho_1\sigma_1,\cdots,\alpha_i\cdots\rho_i\sigma_i}(M_1,\cdots,M_i)\frac{\partial x_1^{\alpha_1}}{\partial u_{11}}\cdots\frac{\partial x_1^{\rho_1}}{\partial u_{1q}}X^{\sigma_1}_{(1)}$$
$$\cdots\frac{\partial x_i^{\alpha_i}}{\partial u_{i1}}\cdots\frac{\partial x_i^{\rho_i}}{\partial u_{iq}}X^{\sigma_i}_{(i)}\delta u_{11}\cdots\delta u_{1q}\delta\tau_1\cdots\delta u_{i1}\cdots\delta u_{iq}\delta\tau_i$$

for all r-dimensional manifolds generated by the path curves of (6.1) (with P_r as their images in the u, τ space) does not necessarily imply conditions (b) of (9.2).

10. **Some geometrical considerations.** In this paragraph we shall give a geometrical interpretation‡ of the process given in the preceding paragraph

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* This can be shown by long straightforward calculations; Goursat follows such a process in the special case of integral invariants. We shall give a much shorter proof for the general problem.

† See Goursat, *Quelques points de la théorie des invariants intégraux*, Journal de Mathématiques, (7), vol. 1 (1915), pp. 246–247.

‡ As a special case, of course, we get Goursat's treatment of integral invariants. Cf. his *Leçons sur le Problème de Pfaff*, §62.

that yielded an attached from a special* invariant functional. The special invariant functionals have additive properties for certain manifolds; it is precisely this property which makes it possible to get an attached from a special invariant functional.

Suppose then

$$\int_{D_r}^{(i)} T(i)\delta x \tag{10.1}$$

is a special invariant functional. Let W_{r-1} be a manifold of $r-1$ dimensions that is not generated by the path curves of the group (6.1). Each point a_0 of W_{r-1} is taken contemporaneously, say for $\tau=0$ (for the convenience of exposition we have the kinematical picture in mind). Each point a_0 of W_{r-1} is taken along the path curves of (6.1) and moved up to a point $a_{\tau'}$, while τ varies from 0 to τ'. The points $a_{\tau'}$ constitute a manifold W'_{r-1}. Let W_r be the manifold generated by the path curves which issue from the points of W_{r-1} and terminate at the corresponding points of W'_{r-1} Let P_{r-1} be the manifold which is generated by the path curves issuing from the points of W_{r-2}, the boundary of W_{r-1}, and terminating at the corresponding points of W'_{r-2}, the boundary of W'_{r-1}. Suppose then the coördinates of the points a_0 of W_{r-1} expressed as functions of the independent variables $u_1, u_2, \cdots, u_{r-1}$ so that to each point of W_{r-1} corresponds a point in a domain $D_{r-1}(0)$ of the u space and conversely. Hence in W_r each point will be a function of $u_1, u_2, \cdots, u_{r-1}, \tau$, and at each point of W_r we shall have

$$\frac{\partial x^\alpha}{\partial \tau} = X^\alpha(M) \qquad (\alpha = 1,2,\cdots,n). \tag{10.2}$$

Our functional (10.1) therefore can be written as

$$\begin{aligned}\int_0^{\tau'}\delta\tau_1\int_0^{\tau'}\delta\tau_2\cdots\int_0^{\tau'}\delta\tau_i\int_{D_{r-1}(\tau_1)}\int_{D_{r-1}(\tau_2)} \\ \cdots\int_{D_{r-1}(\tau_i)} T_{\alpha_1\cdots\rho_1,\cdots,\alpha_i\cdots\rho_i}(M_1, \\ \cdots, M_i)\delta_{11}x_1^{\alpha_1}\delta_{12}x_1^{\beta_1}\cdots\delta_{1r-1}x_1^{\rho_1}\cdots\delta_{i1}x_i^{\alpha_i}\cdots\delta_{ir-1}x_i^{\rho_i}\end{aligned} \tag{10.3}$$

where the multiple tensor $T_{\alpha_1\cdots\rho_1,\cdots,\alpha_i\cdots\rho_i}$ is defined as in (9.8).

---

* By a special invariant functional of r-dimensional manifolds (5.4) we shall understand a functional that satisfies the special conditions for invariance (9.6).

In the above expression we have written $D_{r-1}(\tau)$ to show the dependence of the domain D_{r-1} on $\tau(D_{r-1}(0)$ corresponding to $W_{r-1})$. The ith derivative of (10.3) with respect to τ' at $\tau'=0$ yields

$$(10.4) \qquad i! \int_{D_{r-1}(0)}^{(i)} T(i)\delta x.$$

Let τ^* be a continuous function of position of the points of a_0 of W_{r-1}. On the path curve issuing from the general point a_0 of W_{r-1} consider points a_{τ^*} and $a_{\tau'+\tau^*}$ corresponding to the position of the point a_0 at times $\tau=\tau^*$ and $\tau=\tau^*+\tau'$ respectively. Thus corresponding to $\tau=0$, $\tau=\tau'$, $\tau=\tau^*$, $\tau=\tau^*+\tau'$ there are four manifolds W_{r-1}, W'_{r-1}, W^*_{r-1}, $W^{*\prime}_{r-1}$ whose representative points are a_0, $a_{\tau'}$, a_{τ^*}, $a_{\tau^*+\tau'}$ respectively. W'_{r-1} is the transform of W_{r-1} and $W^{*\prime}_{r-1}$ of W^*_{r-1}. Thus W^*_{r-1} and $W^{*\prime}_{r-1}$ are sections of the tube of path curves whose points are not contemporaneous while those of W_{r-1} and W'_{r-1} are contemporaneous. Evidently then by varying the function τ we get various non-contemporaneous $(r-1)$-dimensional sections of the tube of path curves. The manifold W^*_r will be defined as that bounded by W^*_{r-1} and $W^{*\prime}_{r-1}$; and P^*_{r-1}, as the manifold which is constituted of the path curves issuing from the points of W^*_{r-2}, the boundary of W^*_{r-1}, and terminating at the corresponding points of $W^{*\prime}_{r-2}$, the boundary of $W^{*\prime}_{r-1}$. Furthermore, let w_r be the manifold enclosed by W_{r-1}, W^*_{r-1} and the path curves issuing from W_{r-2} to W^*_{r-2}; and w^*_r, the manifold bounded by W'_{r-1}, $W^{*\prime}_{r-1}$ and the path curves issuing from W'_{r-2} to $W^{*\prime}_{r-2}$. Clearly, the manifold w^*_r is the transform of w_r by the group (6.1). We have

$$(10.5) \qquad W_r^* = W_r + w_r^* - w_r.$$

Employing this relation the following steps in the reasoning are clear in the case of a *special* invariant functional of r-dimensional manifolds:

$$\int_{W_r^*}\int_{W_r^*}\overset{\cdots}{_{(i)}}\int_{W_r^*} = \int_{W_r^*}\int_{W_r^*}\cdots\int_{W_r^*}\int_{W_r+w_r^*-w_r}$$

$$= \int_{W_r^*}\cdots\int_{W_r^*}\int_{W_r} + \int_{W_r^*}\cdots\int_{W_r^*}\int_{w_r^*} - \int_{W_r^*}\cdots\int_{W_r^*}\int_{w_r}$$

$$= \int_{W_r^*}\cdots\int_{W_r^*}\int_{W_r}.$$

Repeating this procedure $i-1$ more times we shall get finally

$$(10.6) \qquad \int_{W_r^*}\overset{\cdots}{_{(i)}}\int_{W_r^*} = \int_{W_r}\overset{\cdots}{_{(i)}}\int_{W_r}.$$

Hence the ith derivative with respect to τ' for $\tau'=0$ of both sides of (10.6) must be equal. This then means, by (10.4), that

$$\int_{D_{r-1}(\tau^*)}^{(i)} T(i)\delta x = \int_{D_{r-1}(0)}^{(i)} T(i)\delta x \tag{10.7}$$

where $D_{r-1}(0)$ corresponds to a given arbitrary contemporaneous section of the tube of path curves while $D_{r-1}(\tau^*)$ corresponds, in general, to a non-contemporaneous section. But the function τ^* is arbitrary, and therefore, in virtue of (10.7), the functional of $(r-1)$-dimensional manifolds

$$\int_{D_{r-1}}^{(i)} T(i)\delta x \tag{10.8}$$

has a constant value for all $(r-1)$-dimensional sections (that are not generated by the path curves), whether contemporaneous or not, of any given but arbitrary tube of path curves. But the multiple tensor T in (10.8) is precisely given by (9.8). Hence we have given a geometrical interpretation of the method, given in §9, that enabled us to step down† from a special invariant functional to an invariant functional that is attached to the path curves of (6.1).

11. **Invariant functionals of closed manifolds.** The invariant theory of functionals of closed manifolds can easily be referred to the invariant theory of functionals of open manifolds. A functional of closed $(r-1)$-dimensional manifolds

$$\int_{D_{r-1}}^{(i)} T(i)\delta x \tag{11.1}$$

can be thrown into the form of a functional of open r-dimensional manifolds

$$\int_{D_r}^{(i)} T(i)\delta x \tag{11.2}$$

by applying Stokes's theorem i times. Thus by applying our previous theory to the functional (11.2) we can get necessary and sufficient conditions for the invariance of (11.1).

We can, however, *start* with a functional (11.2) of open r-dimensional manifolds and then impose the condition that it depend only on the bounding $(r-1)$-dimensional manifolds. Since $R_{\alpha_1\cdots\sigma_1\omega,\alpha_2\cdots\sigma_2,\cdots,\alpha_i\cdots\sigma_i}(M_1, M_2, \cdots,$

---

† This procedure can be considerably generalized when the totality of path curves of the group (6.1) are left invariant by known infinitesimal transformations. I propose to take up these considerations elsewhere.

M_i) has the skew-symmetric properties discussed in §4, it follows that the conditions

$$(11.3) \qquad R_{\alpha_1 \cdots \sigma_1 \omega, \alpha_2 \cdots \sigma_2, \cdots, \alpha_i \cdots \sigma_i}(M_1, M_2, \cdots, M_i) = 0$$
$$(\alpha_1, \cdots, \sigma_1, \omega, \alpha_2, \cdots, \sigma_i = 1, 2, \cdots, n)$$

are sufficient* to insure the dependence of (11.2) only on the bounding $(r-1)$-dimensional manifold.

In order that a functional (11.2) of closed $(r-1)$-dimensional manifolds which satisfies conditions (11.3) be an invariant functional attached to the path curves of (6.1), it is necessary and sufficient that the linear homogeneous algebraic equations

$$T_{\omega\beta_1 \cdots \sigma_1, \alpha_2\beta_2 \cdots \sigma_2, \cdots, \alpha_i \cdots \sigma_i}(M_1, M_2, \cdots, M_i) \overset{\omega}{X}_{(1)} = 0$$
$$(\beta_1, \cdots, \sigma_i = 1, 2, \cdots, n)$$

be satisfied by the multiple tensor $T_{\alpha_1 \cdots \sigma_1, \cdots, \alpha_i \cdots \sigma_i}(M_1, \cdots, M_i)$.

12. **Complete invariant functionals of manifolds.** Up to this point we have been engaged in generalizations of the Poincaré integral invariant. It is possible however to generalize Cartan's† complete integral invariants. Goursat‡ has shown how Cartan's complete integral invariants are bound up with certain Poincaré integral invariants that are attached to the path curves of differential equations. As a matter of fact Cartan's complete integral invariants are special Poincaré integral invariants attached to the path curves of the one-parameter group

$$(12.1) \qquad \frac{dx^1}{X^1(M)} = \frac{dx^2}{X^2(M)} = \cdots = \frac{dx^n}{X^n(M)} = \frac{d\tau}{1} = d\psi$$

in the space time continuum $(x^1, x^2, \cdots, x^n, \tau)$. The parameter of the group (12.1) is ψ and M is a point of the n-dimensional space $(x^1, x^2, \cdots, x^n)$.

Cartan has given a method by means of which one can write down a complete integral invariant corresponding to every Poincaré integral invariant. For example, if

$$\int T_\alpha \delta x^\alpha$$

---

* For functionals of closed curves in three dimensions, cf. Cornelia Fabri, *Sopra alcune proprietà generali delle funzioni che dipendono da altre funzioni e da linee*, Atti della Reale Accademia delle Scienze di Torino, vol. 25, pp. 671.

† Cf. Cartan, *Leçons sur les Invariants Intégraux.*

‡ Comptes Rendus, vol. 174 (1922), pp. 1089–1091.

is an absolute integral invariant of

$$\frac{dx^i}{d\tau} = X^i(M) \qquad (i = 1, 2, \cdots, n) \tag{12.2}$$

then

$$\int T_\alpha \delta x^\alpha - T_\alpha X^\alpha \delta\tau$$

is an absolute complete integral invariant.

It is always possible to pass from a Poincaré integral invariant to a corresponding Cartan complete integral invariant. Such, however, is not, in general, the state of affairs when one considers non-additive functional invariants.

In summations it is convenient to replace the variable τ by x^{n+1}. We can rewrite (12.1) therefore in the form

$$\frac{dx^{\alpha'}}{d\psi} = X^{\alpha'} \qquad (\alpha' = 1, 2, \cdots, n+1). \tag{12.3}$$

In (12.3) the contravariant vector $X^{\alpha'}$ has components $X^1(x^1, x^2, \cdots, x^n)$, $X^2(x^1, \cdots, x^n), \cdots, X^n(x^1, \cdots, x^n)$, 1. We shall use primed letters for indices when the range of the indices is $1, 2, \cdots, n+1$; and unprimed, when the range is 1 to n.

Consider the functional of r-dimensional manifolds in the space continuum $(x^1, x^2, \cdots, x^n)$

$$\int_{D_r}^{(i)} T(i)\delta x. \tag{12.4}$$

Associated with the functional (12.4) we shall define a particular functional of r-dimensional manifolds in the space-time continuum $(x^1, \cdots, x^n, x^{n+1})$ in the following manner.

Let

$$\Lambda_{\alpha_1' \cdots \sigma_1', \alpha_2' \cdots \sigma_2', \cdots, \alpha_i' \cdots \sigma_i'} \qquad (\alpha_1', \cdots, \sigma_i' = 1, 2, \cdots, n+1)$$

be a multiple tensor whose components are given as follows:

$$\Lambda_{\alpha_1 \cdots \sigma_1, \alpha_2 \cdots \sigma_2, \cdots, \alpha_i \cdots \sigma_i} = T_{\alpha_1 \cdots \sigma_1, \alpha_2 \cdots \sigma_2, \cdots, \alpha_i \cdots \sigma_i},$$

$$\begin{aligned} &\Lambda_{\alpha_1 \cdots \sigma_1, \cdots, *(n+1)j_1*, \cdots, *(n+1)j_2*, \cdots, *(n+1)j_k*, \cdots, \alpha_i \cdots \sigma_i} \\ &= (-1)^k T_{\alpha_1 \cdots \sigma_1, *\omega_{j_1}*, \cdots, *\omega_{j_2}*, \cdots, *\omega_{j_k}*, \alpha_i \cdots \sigma_i} X^{\omega_{j_1}}_{(j_1)} X^{\omega_{j_2}}_{(j_2)} \cdots X^{\omega_{j_k}}_{(j_k)}, \end{aligned} \tag{12.5}$$

$$\Lambda_{\alpha_1' \cdots \sigma_1', \cdots, \alpha_i' \cdots \sigma_i'} = 0,$$

whenever two or more subscripts, in any one of the i sets $\alpha_1' \cdots \sigma_1', \cdots, \alpha_i' \cdots \sigma_i'$, take on the value $n+1$.

The stars $\star$ indicate undetermined indices which may have any value from 1 to n.

The functional of r-dimensional manifolds in the space-time continuum $(x^1, x^2, \cdots, x^{n+1})$

$$\int_{\mathfrak{D}_r}\int_{\mathfrak{D}_r} \underset{(i)}{\cdots} \int_{\mathfrak{D}_r} \Lambda_{\alpha_1' \cdots \sigma_1', \cdots, \alpha_i' \cdots \sigma_i'} \delta_{11} x_1^{\alpha_1'} \cdots \delta_{ir} x_i^{\sigma_i'} \tag{12.6}$$

is precisely the functional formed by putting

$$\delta_{lm} x_p^q - X_p^q \delta_{lm} x_p^{n+1} \text{ for } \delta_{lm} x_p^q \qquad \text{in (12.4)}.$$

By calculation one finds that necessary and sufficient conditions that (12.6) be a functional attached to the path curves of the one-parameter group (12.3) are that the equations (9.6) hold. This is verified by employing conditions (9.2) corresponding to the multiple tensor Λ and noting that conditions (b) of (9.2) for $\Lambda_{\alpha_1' \cdots \sigma_1', \cdots, \alpha_i' \cdots \sigma_i'}$ are automatically satisfied. The only conditions on $T_{\alpha_1 \cdots \sigma_1, \cdots, \alpha_i \cdots \sigma_i}$ are therefore conditions (a) of (9.2) for $\Lambda_{\alpha_1' \cdots \sigma_1', \cdots, \alpha_i' \cdots \sigma_i'}$. But these relations reduce to equations (9.6).

To make the argument clear I shall give the details explicitly in the case of a quadratic functional of two-dimensional manifolds

$$\int_{D_2}\int_{D_2} T_{\alpha_1\beta_1, \alpha_2\beta_2}(M_1, M_2) \frac{\partial x_1^{\alpha_1}}{\partial u_{11}} \frac{\partial x_1^{\beta_1}}{\partial u_{12}} \frac{\partial x_2^{\alpha_2}}{\partial u_{21}} \frac{\partial x_2^{\beta_2}}{\partial u_{22}} du_{11} du_{12} du_{21} du_{22}. \tag{12.7}$$

We consider the functional

$$\begin{aligned} &\int_{\mathfrak{D}_2}\int_{\mathfrak{D}_2} T_{\alpha_1\beta_1, \alpha_2\beta_2}(M_1, M_2)\left(\frac{\partial x_1^{\alpha_1}}{\partial u_{11}} - X_{(1)}^{\alpha_1} \frac{\partial x_1^{n+1}}{\partial u_{11}}\right)\left(\frac{\partial x_1^{\beta_1}}{\partial u_{12}}\right. \\ &\qquad \left. - X_{(1)}^{\beta_1} \frac{\partial x_1^{n+1}}{\partial u_{12}}\right)\left(\frac{\partial x_2^{\alpha_2}}{\partial u_{21}} - X_{(2)}^{\alpha_2} \frac{\partial x_2^{n+1}}{\partial u_{21}}\right)\left(\frac{\partial x_2^{\beta_2}}{\partial u_{22}}\right. \\ &\qquad \left. - X_{(2)}^{\beta_2} \frac{\partial x_2^{n+1}}{\partial u_{22}}\right) du_{11} du_{12} du_{21} du_{22}. \end{aligned} \tag{12.8}$$

We can write (12.8) in the form

$$\int_{\mathfrak{D}_2}\int_{\mathfrak{D}_2} \Lambda_{\alpha_1'\beta_1', \alpha_2'\beta_2'} \frac{\partial x_1^{\alpha_1'}}{\partial u_{11}} \frac{\partial x_1^{\beta_1'}}{\partial u_{12}} \frac{\partial x_2^{\alpha_2'}}{\partial u_{21}} \frac{\partial x_2^{\beta_2'}}{\partial u_{22}} du_{11} du_{12} du_{21} du_{22}, \tag{12.9}$$

where

$$
(12.10)\qquad
\begin{aligned}
&\Lambda_{\alpha_1\beta_1,\alpha_2\beta_2} = T_{\alpha_1\beta_1,\alpha_2\beta_2}, &&\Lambda_{n+1\beta_1,n+1\beta_2} = T_{\alpha_1\beta_1,\alpha_2\beta_2}X^{\alpha_1}_{(1)}X^{\alpha_2}_{(2)},\\
&\Lambda_{n+1\beta_1,\alpha_2\beta_2} = -\ T_{\alpha_1\beta_1,\alpha_2\beta_2}X^{\alpha_1}_{(1)}, &&\Lambda_{n+1\beta_1,\alpha_2 n+1} = T_{\alpha_1\beta_1,\alpha_2\beta_2}X^{\alpha_1}_{(1)}X^{\beta_2}_{(2)},\\
&\Lambda_{\alpha_1 n+1,\alpha_2\beta_2} = -\ T_{\alpha_1\beta_1,\alpha_2\beta_2}X^{\beta_1}_{(1)}, &&\Lambda_{\alpha_1\ n+1,n+1\beta_2} = T_{\alpha_1\beta_1,\alpha_2\beta_2}X^{\beta_1}_{(1)}X^{\alpha_2}_{(2)},\\
&\Lambda_{\alpha_1\beta_1,n+1\beta_2} = -\ T_{\alpha_1\beta_1}X^{\alpha_2}_{(2)}, &&\Lambda_{\alpha_1\ n+1,\alpha_2\ n+1} = T_{\alpha_1\beta_1,\alpha_2\beta_2}X^{\beta_1}_{(1)}X^{\beta_2}_{(2)},\\
&\Lambda_{\alpha_1\beta_1,\alpha_2 n+1} = -\ T_{\alpha_1\beta_1,\alpha_2\beta_2}X^{\beta_2}_{(2)}, &&\Lambda_{n+1\ n+1,\alpha_2'\beta_2'} = 0,
\end{aligned}
$$

$$
\Lambda_{\alpha_1'\beta_1',n+1\ n+1} = 0.
$$

In order that (12.9) be an invariant functional attached to the path curves of the one-parameter group (12.3) it is necessary and sufficient that

$$
(12.11)\qquad
\begin{aligned}
&\text{(a)}\quad X^{\omega'}_{(1)}\left(\frac{\partial\Lambda_{\alpha_1'\beta_1',\alpha_2'\beta_2'}}{\partial x_1^{\omega'}}-\frac{\partial\Lambda_{\omega'\beta_1',\alpha_2'\beta_2'}}{\partial x_1^{\alpha_1'}}-\frac{\partial\Lambda_{\alpha_1'\omega',\alpha_2'\beta_2'}}{\partial x_1^{\beta_1'}}\right)=0,\\
&\text{(b)}\quad X^{\omega'}_{(1)}\ \Lambda_{\omega'\beta_1',\alpha_2'\beta_2'} = 0.
\end{aligned}
$$

We have made use of the fact that

$$
\Lambda_{\alpha_1'\beta_1',\alpha_2'\beta_2'}(M_1,M_2) = \Lambda_{\alpha_2'\beta_2',\alpha_1'\beta_1'}(M_2,M_1)
$$

and that $\Lambda_{\alpha_1'\beta_1',\alpha_2'\beta_2'}$ is alternating separately in each of the two sets α_1',β_1' and α_2',β_2'.

The conditions (b) of (12.11) are automatically satisfied; for, from (12.10) we have

$$
(12.12)\qquad
\begin{aligned}
&X^{\omega'}_{(1)}\ \Lambda_{\omega'\beta_1,\alpha_2\beta_2} = X^{\omega}_{(1)}\Lambda_{\omega\beta_1,\alpha_2\beta_2} + \Lambda_{n+1\beta_1,\alpha_2\beta_2} \equiv 0,\\
&X^{\omega'}_{(1)}\ \Lambda_{\omega'\beta_1,n+1\beta_2} = X^{\omega}_{(1)}\Lambda_{\omega\beta_1,n+1\beta_2} + \Lambda_{n+1\beta_1,n+1\beta_2} \equiv 0,\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad.
\end{aligned}
$$

Since

$$
(12.13)\qquad \frac{\partial\Lambda_{\alpha_1'\beta_1',\alpha_2'\beta_2'}}{\partial x_1^{n+1}} = 0,
$$

we can write (a) of (12.11) as

$$
(12.14)\qquad
\begin{aligned}
&X^{\omega}_{(1)}\left(\frac{\partial\Lambda_{\alpha_1'\beta_1',\alpha_2'\beta_2'}}{\partial x_1^{\omega}}-\frac{\partial\Lambda_{\omega\beta_1',\alpha_2'\beta_2'}}{\partial x_1^{\alpha_1'}}-\frac{\partial\Lambda_{\alpha_1'\omega,\alpha_2'\beta_2'}}{\partial x_1^{\beta_1'}}\right)\\
&\qquad+\frac{\partial\Lambda_{\beta_1'n+1,\alpha_2'\beta_2'}}{\partial x_1^{\alpha_1'}}-\frac{\partial\Lambda_{\alpha_1'n+1,\alpha_2'\beta_2'}}{\partial x_1^{\beta_1'}} = 0.
\end{aligned}
$$

If $\alpha_1', \beta_1', \alpha_2', \beta_2' \neq n+1$ in (12.14) we have

$$X^{\omega}_{(1)}\left(\frac{\partial T_{\alpha_1\beta_1,\alpha_2\beta_2}}{\partial x_1{}^{\omega}} - \frac{\partial T_{\omega\beta_1,\alpha_2\beta_2}}{\partial x_1{}^{\alpha_1}} - \frac{\partial T_{\alpha_1\omega,\alpha_2\beta_2}}{\partial x_1{}^{\beta_1}}\right) + \frac{\partial(T_{\alpha_1\omega,\alpha_2\beta_2}X^{\omega}_{(1)})}{\partial x_1{}^{\beta_1}} \tag{12.15}$$
$$- \frac{\partial(T_{\beta_1\omega,\alpha_2\beta_2}X^{\omega}_{(1)})}{\partial x_1^{\alpha_1}} = 0.$$

The other conditions arising from (12.14) are consequences of (12.15), (12.13) and of (12.10). For example, if $\alpha_1' = n+1$; $\alpha_2', \beta_1', \beta_2' \neq n+1$, we have

$$X^{\omega}_{(1)}\left(\frac{\partial(T_{\alpha_1\omega,\alpha_2\beta_2}X^{\alpha_1}_{(1)})}{\partial x_1{}^{\beta_1}} - \frac{\partial(T_{\alpha_1\beta_1,\alpha_2\beta_2}X^{\alpha_1}_{(1)})}{\partial x_1{}^{\omega}}\right) = 0.$$

If $\alpha_1' = n+1$, $\alpha_2' = n+1$; $\beta_1', \beta_2' \neq n+1$, we have

$$X^{\omega}_{(1)}\left(\frac{\partial(T_{\alpha_1\beta_1,\alpha_2\beta_2}X^{\alpha_1}_{(1)}X^{\alpha_2}_{(2)})}{\partial x_1{}^{\omega}} - \frac{\partial(T_{\alpha_1\omega,\alpha_2\beta_2}X^{\alpha_1}_{(1)}X^{\alpha_2}_{(2)})}{\partial x_1{}^{\beta_1}}\right) = 0.$$

The reasoning which we have carried through for the case of the quadratic functional (12.7) of two-dimensional manifolds is directly extensible to a functional (12.4) of r-dimensional manifolds. We can therefore embody the results of this paragraph in the following important theorem.

THEOREM. *A necessary and sufficient condition that the functional* (12.6) *of r-dimensional manifolds in* $n+1$ *dimensions be an invariant functional attached to the path curves of the one-parameter group* (12.3) *is that the functional* (12.4) *of r-dimensional manifolds in n dimensions be a special* invariant functional of the one-parameter group*

$$\frac{dx^i}{dx^{n+1}} = x^i \qquad (i = 1,2, \cdots, n) \tag{12.16}$$

with x^{n+1} *as the parameter of the group.*

As a special case of the above theories we get obviously one of Cartan's† results which states that a complete integral invariant can always be formed from an integral invariant in the sense of Poincaré.

* See third footnote in §10.

† Cf. his *Leçons sur les Invariants Intégraux*, pp. 28–29.

13. **Functionals admitting s-parameter groups.** In Lie's theories of finite continuous groups an s-parameter group of transformations whose infinitesimal transformations have the symbols

$$U^{(j)}f = {}^{(j)}X^{\omega}(M)\frac{\partial f}{\partial x^{\omega}} \qquad (j = 1,2, \cdots, s) \tag{13.1}$$

satisfy the identities

$$(U^{(\alpha)}, U^{(\beta)})f = c_{\alpha\beta\gamma}U^{(\gamma)}f \qquad (\alpha,\beta,\gamma = 1,2, \cdots, s). \tag{13.2}$$

In (13.2), $(U^{(\alpha)}, U^{(\beta)})$ is the alternant operator of Poisson and $c_{\alpha\beta\gamma}$ are the structural constants of the group.

A necessary and sufficient condition that a functional (5.4) of r-dimensional manifolds admit* the s-parameter group (13.1) is that it admit each one of the s independent infinitesimal transformations of the group. Consequently with the aid of conditions (8.4) we see that the invariance conditions are

$$\begin{gathered}\sum_{\lambda=1}^{i}[R^{(\lambda)}_{\alpha_1\cdots\sigma_1,\cdots,\alpha_i\cdots\sigma_i}(M_1, \cdots, M_i) \\ + S^{(\lambda)}_{\alpha_1\cdots\sigma_1,\cdots,\alpha_i\cdots\sigma_i}(M_1, \cdots, M_i)]_j = 0 \\ (\alpha_1, \cdots, \sigma_i = 1,2, \cdots, n; j = 1,2, \cdots, s).\end{gathered} \tag{13.3}$$

In (13.3) the notation $[R^{(\lambda)}_{\cdots}+S^{(\lambda)}_{\cdots}]_j$ is used to denote the fact that ${}^{(j)}X^{\omega}_{(k)}$ has been put in the place of $X^{\omega}_{(k)}$ in $[R^{(\lambda)}_{\cdots}+S^{(\lambda)}_{\cdots}]$. With this notation we can write immediately the necessary and sufficient conditions that a functional (5.4) of r-dimensional manifolds be an invariant functional attached† to the path curves of each one of the s independent one-parameter groups defined by the s independent infinitesimal transformations of the group (13.1). These conditions are

$$\begin{gathered}[R^{(\lambda)}_{\alpha_1\cdots\sigma_1,\cdots,\alpha_i\cdots\sigma_i}(M_1, \cdots, M_i)]_j = 0, \\ [S_{\alpha_1\cdots\rho_1\sigma_1,\cdots,\alpha_k\cdots\rho_k *,\cdots,\alpha_i\cdots\rho_i\sigma_i}(M_1, \cdots, M_k, \cdots, M_i)]_j = 0 \\ (\alpha_1, \cdots, \sigma_i = 1,2, \cdots, n; k,\lambda = 1,2, \cdots, i; j = 1,2, \cdots, s).\end{gathered} \tag{13.4}$$

There is one class of s-parameter groups for which we can affirm the existence of functional invariants corresponding to conditions (13.3) or to attached functionals corresponding to conditions (13.4). This is the case in

* This follows essentially from the familiar reasoning given in Lie's theories of invariant point *functions*.

† See §9.

which the structural constants $c_{\alpha\beta\gamma}$ of the group are all zero* and the number s of essential parameters is at most equal to the number n of dimensions. For such abelian groups, it is always possible* to find a coördinate transformation (x) to (y) such that in the (y) coördinates the s contravariant vectors ${}^{(i)}X^{\omega}$ of the group have the components $\delta^{(j)\omega}$, where

$$\begin{aligned} \delta^{(j)\omega} &= 0 \text{ if } j \neq \omega \\ &= 1 \text{ if } j = \omega \qquad (\omega = 1, 2, \cdots, n;\ j = 1, 2, \cdots, s). \end{aligned}$$

Observing therefore the conditions (13.3) and (13.4) in such a canonical set of coördinates (y) we are enabled to state the following two interesting theorems.

THEOREM 1. *There always exist functionals* (5.4) *of r-dimensional manifolds that admit an abelian s-parameter group* $(s \leqq n)$; *and for such a functional the components of the multiple tensor* $T_{\alpha_1 \cdots \sigma_1, \cdots, \alpha_i \cdots \sigma_i}$ *in the canonical coordinate system* (y) *are functions of the cogredient differences* $y_k^{j+1} - y_k^j$ $(j = 1, \cdots, s-1;\ k = 1, 2, \cdots, i)$ *and of the coördinates* $y_k^l (l = s+1, \cdots, n;$ $k = 1, 2, \cdots, i)$.

THEOREM 2. *There always exist invariant functionals* (5.4) *that are attached to the path curves of s independent one-parameter groups of an abelian s-parameter group* $(s \leqq n)$. *The differential form* $T(i)\delta x$ *of such a functional, when expressed in the canonical variables* (y), *depends only on the variables* $y_k^l (l = s+1, s+2, \cdots, n;\ k = 1, 2, \cdots, i)$ *and their differentials.*

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* Such s-parameter groups are called abelian groups. Cf. Bianchi, *Lezioni sulla Teoria dei Gruppi Continui Finiti di Transformazioni*, p. 260.

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